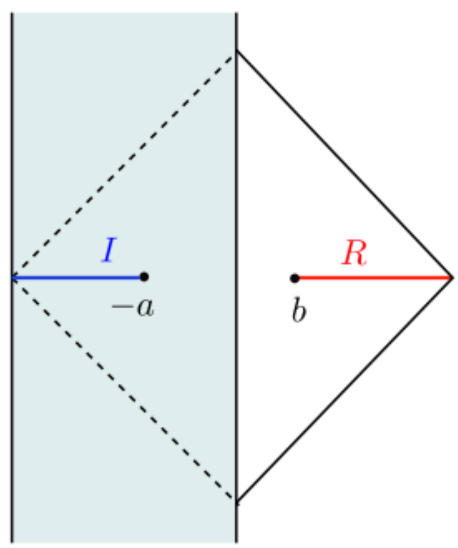
Black Holes and Quantum Info, Fall 2021
Problem Set 6

Due: Dec 17 (optional)

Reading. Do the reading for each lecture posted on the website.

Problem Setup

In this exercise you will work through a simple example of the island effect. The setup is 2d dilaton gravity coupled to a CFT, in a peculiar spacetime which is AdS_2 'glued' to half of 2d Minkowski space. The Penrose diagram and island setup in Lorentzian signature looks like this:



The shaded strip on the left is AdS_2. The half-diamond on the right is half of Minkowski space. There is a CFT that lives everywhere, and gravity that lives only in the shaded region. The metric, in Euclidean signature, is

$$ds^2 = \begin{cases} \frac{4dyd\bar{y}}{(y+\bar{y})^2} & \text{Re } y < -\epsilon_c \\ \frac{1}{\epsilon_c^2} dyd\bar{y} & \text{Re } y > -\epsilon_c \end{cases} \tag{0.1}$$

where $\epsilon_c \to 0$ is a cutoff near the AdS_2 boundary.

To proceed we will need a couple of formulas. First, this is *dilaton* gravity, not Einstein gravity, which means that Newton's constant is promoted to a dynamical field, $\frac{1}{4G} \to \phi$. The action is

$$I_{\text{grav}} = -\frac{1}{4\pi} \int \sqrt{g}\phi(R+2) \ . \tag{0.2}$$

Thus the gravitational entropy, instead of Area/$4G$, is

$$S_{\text{grav}} = \phi \ . \tag{0.3}$$

(Note that in 2d, we are calculating the 'area' of a point. So there is no integral to do.) The gravitational action is included only in the AdS_2 region; there is no gravity and therefore no dilaton in the Minkowski region. The dilaton in the metric (0.1) is

$$\phi = -\frac{2\phi_r}{y+\bar{y}} \ , \tag{0.4}$$

where ϕ_r is a constant. (This comes from solving the dilaton+gravity eom.)

The last ingredient we need is the formula for the entropy of a CFT in curved spacetime. In lecture we derived the formula in flat spacetime, $S = \frac{c}{3} \log \frac{L}{\epsilon}$. In curved spacetime, with the metric written in conformal gauge as

$$ds^2 = e^{2\rho(y,\bar{y})} dy d\bar{y} \tag{0.5}$$

the entanglement entropy of a CFT on the interval $[y_1, y_2]$ is

$$S_{\text{CFT}} = \frac{c}{6} \log \left(\frac{|y_1 - y_2|^2}{\epsilon^2 e^{-\rho(y_1,\bar{y}_1)-\rho(y_2,\bar{y}_2)}} \right) \tag{0.6}$$

Finally, in the y coordinate, define the regions I and R as the intervals

$$I = \{y \in (-\infty, -a]\} \tag{0.7}$$

$$R = \{y \in [b, \infty)\} \tag{0.8}$$

at along the real-y axis (i.e. at time = 0). Think of region R as analogous to the Hawking radiation; I is the island. You can think of the AdS_2 region as a zero-temperature extremal black hole with finite entropy.

Problems

1. Suppose there is no island effect. What is the von Neumann entropy of region R? Argue (in words) that your answer is inconsistent with the principles of black hole thermodynamics.

2. With the regions I, R defined above, show that

$$S(a,b) := S_{\text{grav}}(\partial I) + S_{\text{CFT}}(I \cup R) \tag{0.9}$$

   is given by

$$S(a,b) = \frac{\phi_r}{a} + \frac{c}{6}\log\frac{(a+b)^2}{a} + \text{const} \tag{0.10}$$

3. Extremize $S(a,b)$ to find the location of the quantum extremal surface (ie solve for a).

4. What is the entropy $S(\rho_R)$, according to the island rule? (The formula for a is a little messy, so don't bother trying to simplify the expression.)