To prime or not to prime:

Lecture notes on conformal transformations

Cornell Physics 7661

Tom Hartman, October 2021

In these notes I will explain the relation between coordinate transformations, Poincare transformations, conformal transformations, and Weyl transformations in quantum field theory.

Briefly, coordinate changes are trivial relabelings; Poincare transformations are physical symmetries inherited from isometries of the background manifold; conformal transformations are physical symmetries inherited from conformal isometries of the background manifold; and Weyl transformations are not symmetries at all, but relationships between two different theories on different manifolds. (A 'physical symmetry' relates physically different observables in the same theory, organizes the quantum states of a fixed Hilbert space into representations, etc.)

Summary

Here is a collection of the main formulas we will derive, for easy reference. Many of these formulas are redundant. In this summary $\mathcal{O}(x)$ is a scalar but below I will include the extra factors for spinning operators. Let

$$y = x' = f(x) \ . \tag{1}$$

I label correlation functions by the invariant line element ds^2, or by the metric components $g_{\mu\nu}$, or with the shorthand

$$ds^2 = g(x)dx^2 := g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{2}$$

Coordinate invariance

$$\mathcal{O}'(x') = \mathcal{O}(x) \tag{3}$$

$$\mathcal{O}'(y) = \mathcal{O}(f^{-1}(y)) \tag{4}$$

$$\langle \mathcal{O}'(x_1') \cdots \rangle_{ds^2} = \langle \mathcal{O}(x_1) \cdots \rangle_{ds^2} \tag{5}$$

$$\langle \mathcal{O}(x_1) \cdots \rangle_{g(x)dx^2} = \langle \mathcal{O}(x_1') \cdots \rangle_{g'(x)dx^2} \tag{6}$$

Poincare symmetry

$$\langle \mathcal{O}'(x_1) \cdots \rangle_{ds^2} = \langle \mathcal{O}(x_1) \cdots \rangle_{ds^2} \ . \tag{7}$$

Conformal symmetry

$$\mathcal{O}'(x') = \Omega(x)^{-\Delta} \mathcal{O}(x), \quad \Omega(x) = |\partial x'/\partial x|^{1/d} \tag{8}$$

$$\mathcal{O}'(y) = \tilde{\Omega}(y)^{\Delta} \mathcal{O}(f^{-1}(y)) \ , \quad \tilde{\Omega}(y) = |\partial x/\partial y|^{1/d} = \Omega(x)^{-1} \tag{9}$$

$$\langle \mathcal{O}(x_1) \cdots \rangle_{ds^2} = \langle \mathcal{O}'(x_1) \cdots \rangle_{ds^2} \tag{10}$$

$$\langle \mathcal{O}(x_1) \cdots \rangle_{\Omega^2 ds^2} = \langle \mathcal{O}'(x_1') \cdots \rangle_{ds^2} \ , \qquad \Omega = |\partial x'/\partial x|^{1/d} \tag{11}$$

where $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$ in these expressions and in the last formula, $\Omega^2 ds^2 = g_{\mu\nu}(x') dx'^\mu dx'^\nu$. In $d = 2$, eqn (10) applies only to $SO(2,2)$ transformations.

Weyl transformations

$$\langle \mathcal{O}(x) \cdots \rangle_{\Omega^2 ds^2} = \Omega(x)^{-\Delta} \cdots \langle \mathcal{O}(x) \cdots \rangle_{ds^2} \tag{12}$$

Now we proceed to the details.

Coordinate changes

Under a general coordinate change, tensors transform as

$$\mathcal{O}'_{\mu\nu\cdots}(x') = J_\mu^\alpha J_\nu^\beta \cdots \mathcal{O}_{\alpha\beta\cdots}(x) \ , \quad J_\mu^\alpha = \frac{\partial x^\alpha}{\partial x'^\mu} \ . \tag{13}$$

I will suppress the indices and use the shorthand

$$\mathcal{O}'(x') = R(x,x')\mathcal{O}(x) \ , \quad x' = f(x) \tag{14}$$

so throughout this note you can set $R=1$ for scalars. Equivalently, with $y = x' = f(x)$,

$$\mathcal{O}'(y) = R(x,y)\mathcal{O}(f^{-1}y) \ . \tag{15}$$

Aside: Often people swap the labels $x \leftrightarrow x'$ to write this as $\mathcal{O}'(x) = R(x',x)\mathcal{O}(x')$, with $x' = f^{-1}(x)$. This is common in the literature on CFTs in $d > 2$ dimensions. I will stick to the convention in (14) whenever I use a prime on x'. I prefer this convention because only in this case is the operator $\mathcal{O}'(x')$ actually a local operator supported at the spacetime point x'; the operator $\mathcal{O}'(x_0)$ is supported at the point $x' = x_0$, not $x = x_0$. In other words, the argument of $\mathcal{O}'$ is a value of the coordinate x', even if we happen to call it x... if that was confusing then I've made my point!

The physical information in a correlation function is coordinate invariant. This implies

$$\textbf{coord:} \qquad \langle \mathcal{O}'(x_1') \cdots \rangle_{g'} = R(x_1, x_1')\langle \mathcal{O}(x_1)\cdots\rangle_g \tag{16}$$

where I have labeled the correlator by the metric components $g'_{\mu\nu}$. Both correlators in this equation are of course in the same physical metric, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g'_{\mu\nu}dx'^\mu dx'^\nu$.

For example if we rewrite a CFT 2-point function in spherical coordinates,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = |x_1 - x_2|^{-2\Delta} \Rightarrow \langle \mathcal{O}(r_1,\hat{n}_1)\mathcal{O}(r_2,\hat{n}_2)\rangle = |r_1^2 + r_2^2 - 2r_1r_2\hat{n}_1\cdot\hat{n}_2|^{-\Delta} \tag{17}$$

we are using (16). We often omit the prime on $\mathcal{O}'(r_1,\theta_1)$, as I have done here, because it is implied by the arguments. But clearly there is a secret prime, because if I ask "What is $\mathcal{O}(3,0)$?" you will respond "In what coordinates?" and I will have to tell you or I'm not making sense.

We can also write coordinate changes in a format where the line element changes. For example, a scalar 2-point obviously satisfies

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{dx^2} = \left\langle \mathcal{O}(\frac{x_1}{\lambda})\mathcal{O}(\frac{x_2}{\lambda})\right\rangle_{\lambda^2 dx^2} \tag{18}$$

This is not a dilatation or Weyl transformation; it is a trivial change of units, and the equation holds in any QFT. Similarly for general tensors and general coordinate changes,

$$\textbf{coord}': \qquad R(x_1, x_1')\langle \mathcal{O}(x_1)\cdots\rangle_{g(x)dx^2} = \langle \mathcal{O}(x_1')\cdots\rangle_{g'(x)dx^2} \tag{19}$$

Derivation: (19) is equivalent to

$$\langle \mathcal{O}'(x_1')\cdots\rangle_{ds^2} = \langle \mathcal{O}(f(x_1))\cdots\rangle_{ds_f^2} \tag{20}$$

where the line element ds_f^2 is obtained by replacing $x \to f^{-1}(x)$ in ds^2,

$$ds_f^2 = g_{\mu\nu}(f^{-1}(x))df^{-1}(x)^\mu df^{-1}(x)^\nu \tag{21}$$

(20) is basically trivially. To derive it, starting with the lhs, write the metric as $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu\nu}(f^{-1}(x'))df^{-1}(x')^\mu df^{-1}(x')^\nu$, then simply rename $x' \to x$. The coordinate is a dummy variable so we are free to call it whatever we want. Note, however, that the coordinate x is not really the 'same' coordinate on the two sides of (18) or (19), because the metrics are different.

Symmetries from isometries

So far all we have discussed is a trivial relabeling of points. Now we will see that isometries of the background metric lead to physical symmetries. An isometry is a coordinate transformation such that

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) \ . \tag{22}$$

For example in flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ the isometries are Poincare transformations.

The fact that isometries $\Rightarrow$ symmetries follows immediately from (19). If $x \to x'$ is an isometry, then we can replace $g'(x) = g(x)$ on the rhs of (19), giving

$$R(x_1, x_1')\langle \mathcal{O}(x_1)\cdots\rangle_{ds^2} = \langle \mathcal{O}(x_1')\cdots\rangle_{ds^2} \ . \tag{23}$$

This is an equality between two different physical observables in the same metric, hence a physical symmetry. Setting

$$y = x' = f(x) \tag{24}$$

it becomes

$$\textbf{Poincare:} \quad \langle \mathcal{O}'(y_1)\cdots\rangle_{ds^2} = \langle \mathcal{O}(y_1)\cdots\rangle_{ds^2} \tag{25}$$

Thus correlators are invariant under replacing all operators by

$$\mathcal{O}(y_1) \to \mathcal{O}'(y_1) := R(x_1, y_1)\mathcal{O}(f^{-1}y_1) \ . \tag{26}$$

(And note $R(x_1, y_1) = R(f(x_1), x_1)^{-1}$.) This can also be derived from the action of a conserved charge Q that generates the symmetry,

$$\mathcal{O}'(y) = e^{Q}\mathcal{O}(y)e^{-Q} \ . \tag{27}$$

We have assumed implicitly that the vacuum state is invariant under the isometry, $e^{-Q}|0\rangle = |0\rangle$, so that

$$\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots|0\rangle = \langle 0|e^{Q}O_1(x_1)e^{-Q}e^{Q}\mathcal{O}_2(x_2)e^{-Q}\cdots|0\rangle \ . \tag{28}$$

Infinitessimal symmetries correspond to infinitessimal action of the charge, so with $x' = x - \delta x$,

$$\delta\mathcal{O}(x_1) := \mathcal{O}'(x_1) - \mathcal{O}(x_1) \tag{29}$$

$$= [Q_{\delta x}, \mathcal{O}(x_1)] \tag{30}$$

$$= \mathcal{L}_{\delta x}\mathcal{O}(x_1) \tag{31}$$

where $\mathcal{L}_\zeta$ is the Lie derivative.

Conformal transformations

We will now specialize to CFTs.

A conformal isometry is a coordinate change such that

$$g'_{\mu\nu}(x) \propto g_{\mu\nu}(x) \tag{32}$$

As far as its action on the coordinates, a conformal transformation is a conformal isometry. However its action on the fields is different from a coordinate change.

For the rest of this note, I will assume the manifold is conformally flat, $ds^2 \propto ds^2_{\text{flat}}$. The conformal isometries are coordinate transformations that rescale and rotate at each point,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) S^\mu{}_\nu(x), \quad S^\mu{}_\nu \in SO(d) \ . \tag{33}$$

Derivation: The conformal isometries depend only on the conformal class of the metric (i.e. are insensitive to the overall prefactor) so we can assume $g_{\mu\nu} = \delta_{\mu\nu}$. By the definition of conformal (32), there exists a function $\Omega(x)$ such that

$$g'_{\mu\nu}(x') = \Omega(x)^{-2} \delta_{\mu\nu} \ . \tag{34}$$

Plugging in $g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\delta_{\mu\nu}$, the equation says $MM^T = \mathbf{1}$ with $M^\alpha{}_\mu = \Omega(x)\frac{\partial x^\alpha}{\partial x'^\mu}$. Therefore M is locally a rotation, and inverting gives (33). This derivation assumed x is a Cartesian coordinate, but the result (33) is general because changing coordinates will only turn $S^\mu{}_\nu$ into the rotation matrix of your new coordinates. (It is only literally an $SO(d)$ matrix in Cartesian coordinates; I really mean it's a rotation matrix. In any coordinates, $\det S = 1$.)

Taking the determinant of (33) gives

$$\Omega(x) = \left|\frac{\partial x'}{\partial x}\right|^{1/d} := \left|\det \frac{\partial x'^\mu}{\partial x^\nu}\right|^{1/d} \ . \tag{35}$$

Here is another useful way to characterize conformal transformations. Start with the metric $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, and replace $x \to f(x)$. The result can be written

$$ds^2 \to ds_f^2 = \Omega^2(x)ds^2 \tag{36}$$

if and only if the transformation $x' = f(x)$ is conformal.

This is a practical way to find conformal transformations and to calculate Ω. Note, however, that when we do a conformal transformation in CFT, we do *not* actually act on the metric: ds^2 is fixed.

We will now discuss conformal transformations in a CFT. As I mentioned, they act on the coordinates by conformal isometries, but on fields they do not act like a coordinate change. Unfortunately this means that from now on in this note, $\mathcal{O}'$ means something different: it will now denote the conformal transformation.

The conformal transformation of a spin-ℓ primary operator is

$$\mathcal{O}'(x') = \Omega(x)^{-\Delta+\ell}R(x,x')\mathcal{O}(x) \ , \quad \Omega(x) = \left|\frac{\partial x'}{\partial x}\right|^{1/d} \tag{37}$$

Equivalently, with $y = x' = f(x)$,

$$\mathcal{O}'(y) = \tilde{\Omega}(y)^{\Delta-\ell}R(x,y)\mathcal{O}(x) \ , \quad \tilde{\Omega}(y) = \left|\frac{\partial x}{\partial y}\right|^{1/d} \ . \tag{38}$$

Aside: Often people swap $x \leftrightarrow x'$, ie define $x' = f^{-1}(x)$ and write the conformal transformation as

$$\mathcal{O}'(x) = \left|\frac{\partial x'}{\partial x}\right|^{(\Delta-\ell)/d} R(x',x)\mathcal{O}(x') \tag{39}$$

I will not use this convention; see comments below (15). But for comparison to other references, note that this means my $\tilde{\Omega}$ is equal to the Ω that appears in, for example, Simmons-Duffin's and Rychkov's lectures.

For a dilatation, (37) and says

$$\mathcal{O}'(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x) \tag{40}$$

and (38) says $\mathcal{O}'(x) = \lambda^{\Delta}\mathcal{O}(\lambda x)$.

Finite conformal transformations act as

$$U\mathcal{O}(x)U^{-1} = \mathcal{O}'(x) \tag{41}$$

with $U := e^{Q[\epsilon]}$. Infinitessimally, for scalar operator

$$\delta\mathcal{O}(x) = \mathcal{O}'(x) - \mathcal{O}(x) \tag{42}$$

$$= [Q[\epsilon], \mathcal{O}(x)] \tag{43}$$

$$= \epsilon \cdot \partial\mathcal{O} + \frac{\Delta}{d}(\partial \cdot \epsilon)\mathcal{O} \tag{44}$$

where we set $x' = x + \epsilon(x)$ in (38). The transformation of a spinning operator has additional terms from the rotational part of the conformal transformation.

In a CFT, the vacuum state is invariant under all conformal transformations in $d > 2$, and under the $SO(2,2)$ global subgroup in $d = 2$. Therefore, excluding the non-global 2d transformations, vacuum correlators are invariant under $\mathcal{O}(x) \to \mathcal{O}'(x)$:

$$\textbf{Conformal:} \qquad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\rangle_{ds^2} = \langle \mathcal{O}'(x_1)\mathcal{O}'(x_2)\cdots\rangle_{ds^2} \ . \tag{45}$$

Equivalently,*

$$\textbf{Conformal}':\qquad \langle \mathcal{O}(x)\cdots\rangle_{\Omega^2 g(x)dx^2} = \langle \mathcal{O}'(x')\cdots\rangle_{g(x)dx^2}\ , \qquad \Omega = |\partial x'/\partial x|^{1/d}\ . \tag{49}$$

and note that $\Omega^2 g(x)dx^2 = g(x')(dx')^2$. The symmetries (45) and (49) hold for all operators, not just primaries, but the formula for $\mathcal{O}'$ is different for non-primaries.

The Weyl transformation law is

$$\textbf{Weyl:}\qquad \langle \mathcal{O}(x)\cdots\rangle_{\Omega^2 ds^2} = \Omega(x)^{-\Delta}\cdots\langle \mathcal{O}(x)\cdots\rangle_{ds^2}\ . \tag{50}$$

This holds for any Ω^2, and it generally changes the manifold. It is not a "symmetry" because it relates two different theories, not states or operators in the same theory.

The special Weyl transformations with $\Omega = |\partial x'/\partial x|^{1/d}$ can be reintepreted as conformal transformations. Consider the Weyl transformation law for a conformal change of coordinates $x \to x'$ with

$$(dx')^2 = \Omega^2 dx^2, \quad \Omega = \left|\frac{\partial x'}{\partial x}\right|^{1/d} \tag{51}$$

Then the Weyl transformation (50) reduces to (49).

This causes endless confusion so let's reiterate:

- The conformal transformation $\mathcal{O}(x) \to \mathcal{O}'(x)$ is a symmetry; observables are invariant.
- Weyl invariance (for primaries) is the statement that the operator $\Omega^{-\Delta}\mathcal{O}(x)$ in

*Derivation: In (19) we derived

$$\langle \mathcal{O}(x_1)\cdots\rangle_{dx^2} = \langle \mathcal{O}(f^{-1}(x_1))\cdots\rangle_{df^2} \tag{46}$$

where $df^2 = |\partial f/\partial x|^{1/d}dx^2$ denotes the flat metric in the coordinate f. Evaluate this at $x_i = f(y_i)$,

$$\langle \mathcal{O}(f(y_1))\cdots\rangle_{dx^2} = \langle \mathcal{O}(y_1)\cdots\rangle_{|\partial f/\partial x|^{1/d}dx^2} \tag{47}$$

On the left, do a conformal transformation to replace $\mathcal{O}(f(y_i)) \to \mathcal{O}'(f(y_i))$. On the right, the operator is at $x_1 = y_1$, so we can relabel the coordinate $x \to y$. Thus

$$\langle \mathcal{O}'(f(y_1))\cdots\rangle_{dx^2} = \langle \mathcal{O}(y_1)\cdots\rangle_{df(y)^2} \tag{48}$$

This is (49) with $x' = f(x)$. (There is a probably an easier way to explain this but I couldn't find it....)

the original theory calculates correlators of a related theory in the metric $\Omega^2 ds^2$.

- These two statements overlap in the special case that the Weyl transformation takes the form $\Omega = |\partial x'/\partial x|^{1/d}$.

When people say 'do a conformal transformation', they could mean two different things depending on context. Sometimes they mean $\mathcal{O}(x) \to \mathcal{O}'(x)$, with the metric fixed, which is a symmetry. Other times they mean $\mathcal{O}'(x')$, with the metric fixed, which is not a symmetry, but tells us the physics in the metric $\Omega^2 ds^2$. The latter is related to Weyl transformations, but note that the operator $\mathcal{O}'(x')$ is an operator in the original theory on the original metric.

Example: Physics on the cylinder in 2d CFT

In 2d the cylinder is a conformal transformation of the plane, with the mapping

$$z = e^{iz'} \ . \tag{52}$$

The metrics on the plane and cylinder are

$$ds^2_{\rm pl} = dz d\bar{z} \ , \quad z \in \mathbb{C} \tag{53}$$

$$ds^2_{\rm cyl} = dz' d\bar{z}', \quad z' \sim z' + 2\pi, z' \in \mathbb{C} \tag{54}$$

and they are related as

$$ds^2_{\rm pl} = e^{i(z'-\bar{z}')} dz' d\bar{z}' = \Omega^2 ds^2_{\rm pl} \ . \tag{55}$$

To find the physics on the cylinder, we can use the conformal transformation formula (19). For example the stress tensor on the cylinder has

$$\langle T(z')\rangle_{\rm cyl} = \langle T(z')\rangle_{dz'd\bar{z}'} \tag{56}$$

$$= \langle T'(z')\rangle_{dzd\bar{z}} \tag{57}$$

$$= \langle \left(\frac{\partial z}{\partial z'}\right)^2 \left[T(z) - \frac{c}{12}\{z', z\}\right]\rangle_{dzd\bar{z}} \tag{58}$$

$$= -\frac{c}{12}\left(\frac{\partial z}{\partial z'}\right)^2 \{z', z\} \tag{59}$$

$$= \frac{c}{24} \ . \tag{60}$$

This is the correct Casimir energy density on the cylinder.