9.

Energy Conditions

Read: Poisson §2.1

This is a good time to take a brief tangent to discuss the role of positive energy in classical and quantum gravity.

Classical GR

In classical GR we can assume whatever we please about $T_{\mu\nu}$.

"$T^{tt} > 0$?"

v forward timelike; k forward null

"Strong" (SEC)

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) v^\alpha v^\beta \geq 0$$

$$EE \Rightarrow \quad R_{\alpha\beta} v^\alpha v^\beta \geq 0$$

I can't give any intuition for this - it's often false, as we'll see!

Violated: positive CC (does not focus)

"Weak" (WEC)

$$T_{\alpha\beta} v^\alpha v^\beta \geq 0$$

= energy density according to observer with velocity v

(strong $\not\Rightarrow$ weak!)

This sounds reasonable, but is easily violated.

Violated: Casimir energy, neg. CC

"Null" (NEC)

$$T_{\alpha\beta} n^\alpha n^\beta \geq 0$$

almost true - violated by $\mathcal{O}(\hbar)$ in QFT

(ex: Hawking radiation)

Quantum Energy Positivity

So all these energy conditions are great for proving theorems in GR, but are they true?

No! In fact it is a deep and general fact about quantum fields that there is No localized lower bound on $T_{\mu\nu}$.

$H = \int T_{00} \geq 0$, but

QFT does not satisfy any local energy condition.

Proof: (for T_{00})

$$\langle 0 | T_{00}(x) | 0 \rangle = 0 \quad \text{(Poincaré invariance)}$$

$$\langle 0 | T_{00}(x) T_{00}(y) | 0 \rangle \neq 0$$

Let $|1\rangle = T_{00}(y)|0\rangle$ and find in 2x2 subspace $|0\rangle, |1\rangle$,

$$\langle i | T_{00}(x) | j \rangle = \begin{array}{c c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 0 & b \\ b^* & a \end{pmatrix} \end{array}$$

eigenvalues $\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + |b|^2}$

$\Rightarrow \exists$ state with $\langle \psi_- | T_{00} | \psi_- \rangle < 0$

The same argument applies to the energy of any finite region in space.

QFT does satisfy non-local energy conditions,

ANEC

averaged

$$\int_{\substack{\text{achronal} \\ \text{null ray}}} dn^\alpha \, T_{\alpha\beta} \, n^\beta \geq 0$$

Probably true; "proved" in QFT in Minkowski