3.

Correlators and Analyticity

Read: arXiv 1509.00014 section 3.

(posted on website as "causality review")

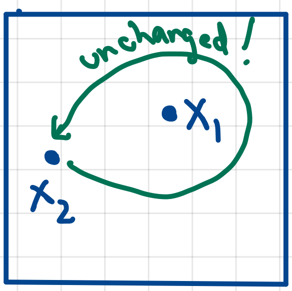
Properties (in $\mathbb{R}^D$)

Euclidean correlators of local operators at separated points are

① independent of ordering

$$\langle [\phi(x_1), \phi(x_2)] \cdots \rangle = 0 \qquad (x_1 \neq x_2)$$

①' single-valued:



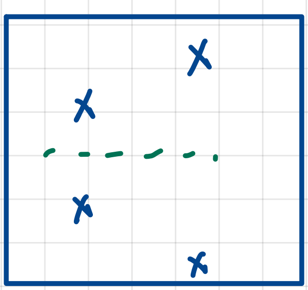
ex: In CFT,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = |x_1 - x_2|^{-2\Delta}$$

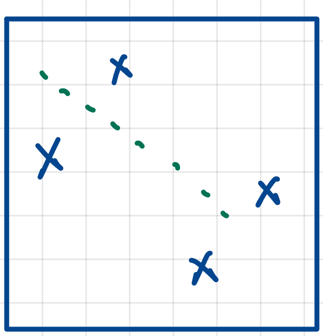
$$= \left[(x_1 - x_2)^2\right]^{-\Delta}$$

no cuts!

② Reflection Positive



≥ 0



≥ 0

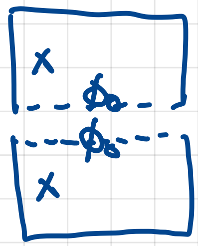
Provisional "defn." of CFT:

a QFT in which we can choose a basis of local operators with definite mass dimension.

"Proof" (This is not so trivial with gauge fields ...)

$$\int D\phi \quad \phi(\tau_1, \vec{x}_1)\,\phi(-\tau_1, \vec{x}_1) \cdots e^{-S_E}$$

$$= \int d\phi_0$$



$$= \int d\phi_0(\vec{x}) \left[\int D\phi \quad \phi(\tau_1, \vec{x}_1)\, e^{-S_E} \right]^2$$

$$\geq 0$$

(assuming $S_E \in \mathbb{R}$ and parity even

or $S_E \in i\mathbb{R}$ and parity odd)

Interpretation

Defn. $\quad \mathcal{O}(\tau, \vec{x})^\dagger = \mathcal{O}^\dagger(-\tau, \vec{x})$

$$= \mathcal{O}(-\tau, \vec{x}) \quad \text{for real field}$$

$$\langle \mathcal{O}(-\tau, \vec{x})\,\mathcal{O}(\tau, \vec{x}) \rangle = \langle 0 | \mathcal{O}(\tau, \vec{x})^\dagger\, \mathcal{O}(\tau, \vec{x}) | 0 \rangle$$

$$= \left\| \mathcal{O}(\tau, \vec{x}) | 0 \rangle \right\|^2 \geq 0$$

The full statement of reflection positivity allows for arbitrary superpositions, I'll leave you to work out the inequality.

Comment:

Some of this carries over to other manifolds, especially if they are highly symmetric.

E.g. all true on S^D.

(3)

$$\int_{x_i \in \mathbb{R}^D} f(x_1, \dots, x_n)\, \phi(x_1) \cdots \phi(x_n) |0\rangle \quad \text{is dense in } \mathcal{H}$$

Roughly means that you can make <u>any state</u> with these operator insertions (by taking superpositions), and that you can therefore learn a <u>lot</u> about a QFT from Lorentzian correlators.

("reconstruction theorems"; many technical assumptions ...)

Lorentzian Correlators ($\mathbb{R}^{1,D-1}$)

Emphasize: "Euclidean QFT" vs. "Lorentzian QFT" are not two different theories. They are two different ways to study the same theory. This is especially powerful in relativistic QFT because often observables are related by an analytic continuation.

ex. CFT 2-point function

$$\langle \mathcal{O}(\tau_1,\vec{x}_1)\,\mathcal{O}(\tau_2,\vec{x}_2)\rangle = \left[(\tau_1-\tau_2)^2 + (\vec{x}_1-\vec{x}_2)^2\right]^{-\Delta}$$

Continue $\tau \to it$

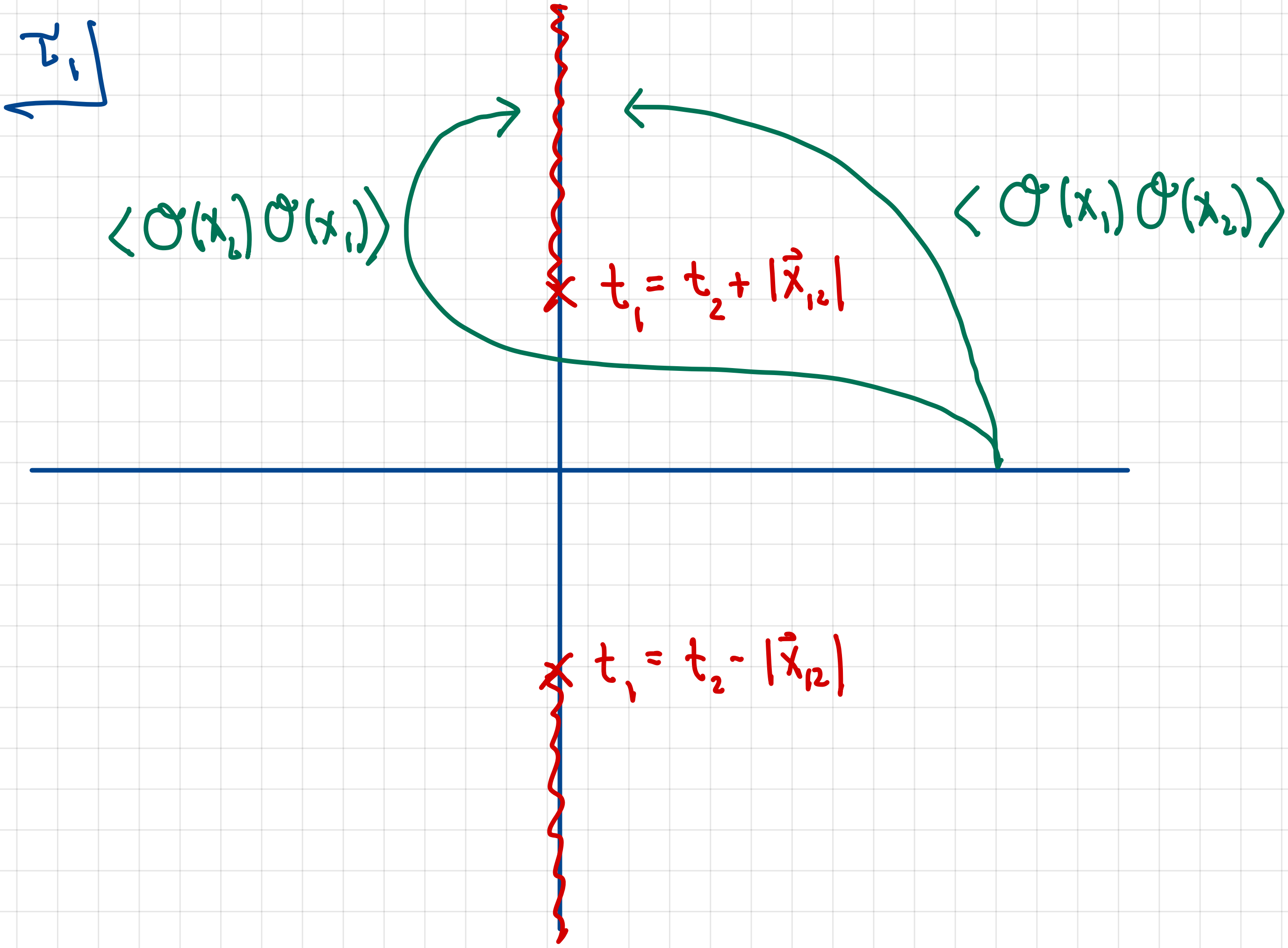
$$\langle \mathcal{O}(t_1,\vec{x}_1)\,\mathcal{O}(t_2,\vec{x}_2)\rangle = \left[-(t_1-t_2)^2 + (\vec{x}_1-\vec{x}_2)^2\right]^{-\Delta}$$

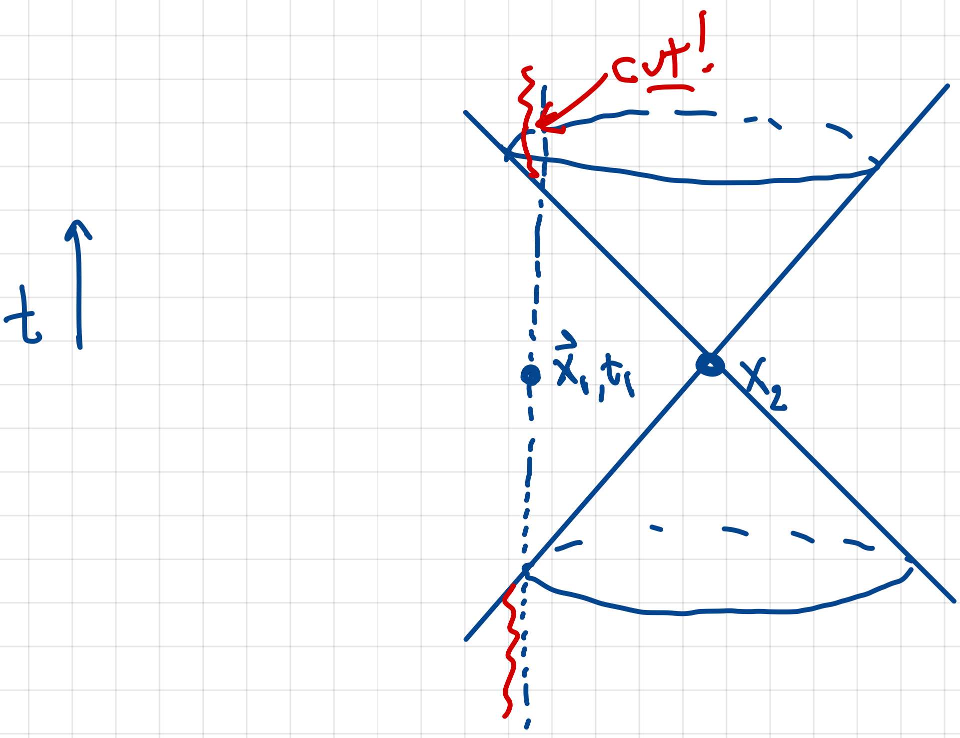
unambiguous at spacelike separation, but for TL $\to$

for $x_{12}^2 < 0$

$$= \left[(t_{12})^2 - |\vec{x}_{12}|^2\right]^{-\Delta} \underbrace{(-1)^{-\Delta}}_{e^{\pm i\pi\Delta}}$$

ambiguity!


τ_1
$\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$
$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$
$t_1 = t_2 + |\vec{x}_{12}|$
$t_1 = t_2 - |\vec{x}_{12}|$


cut!
t
$\vec{x}_1, t_1$
x_2

Now in formulas:

Thus for $t_1 > t_2 + |\vec{x}_{12}|$,

$$\langle \mathcal{O}(t_1, \vec{x}_1)\, \mathcal{O}(t_2, \vec{x}_2) \rangle$$

$$= \left[-(t_{12})^2 + |\vec{x}_{12}|^2 \right]^{-\Delta} \Big|_{\text{path } 1}$$

$$= e^{i\pi\Delta} |x_{12}^2|^{-\Delta}$$

and

$$\langle \mathcal{O}_2 \mathcal{O}_1 \rangle = e^{-i\pi\Delta} |x_{12}^2|^{-\Delta}$$

(explain how this relates to contours)

$i\varepsilon$ shorthand

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \left[-(t_1 - i\varepsilon - t_2)^2 + |\vec{x}_{12}|^2 \right]^{-\Delta}$$

$$\langle \mathcal{O}(x_2)\, \mathcal{O}(x_1) \rangle = \left[-(t_1 - (t_2 - i\varepsilon))^2 + |\vec{x}_{12}|^2 \right]^{-\Delta}$$

w/ principal branch for log.

Which is which?

$\mathrm{Im}\,\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\rangle$ for $\vec{x}_1=\vec{x}_2=0$, $t_1>t_2$

$$= \mathrm{Im}\,\langle 0|\cancel{e^{iHt_1}}\,\mathcal{O}(0)\,e^{-iH(t_1-t_2)}\,\mathcal{O}(0)\,\cancel{e^{-iHt_2}}|0\rangle$$

$$\int d^4p\ \underbrace{\rho(p^2)}_{\text{d.o.s.}}\,\Theta(p^0)\,|p\rangle\langle p|$$

(Poincaré reps)

$$= \mathrm{Im}\int d^4p\ e^{-ip^0(t_1-t_2)}\,\underbrace{\underbrace{\langle\mathcal{O}(0)|p\rangle\langle p|\mathcal{O}(0)\rangle}_{|\langle\mathcal{O}(0)|p\rangle|^2}\,\rho(p^2)\,\Theta(p^0)}_{\geq 0}$$

I.E.,

$$\mathrm{Im}\,G^{(2)}(p) \geq 0 \qquad \text{"spectral density"}$$

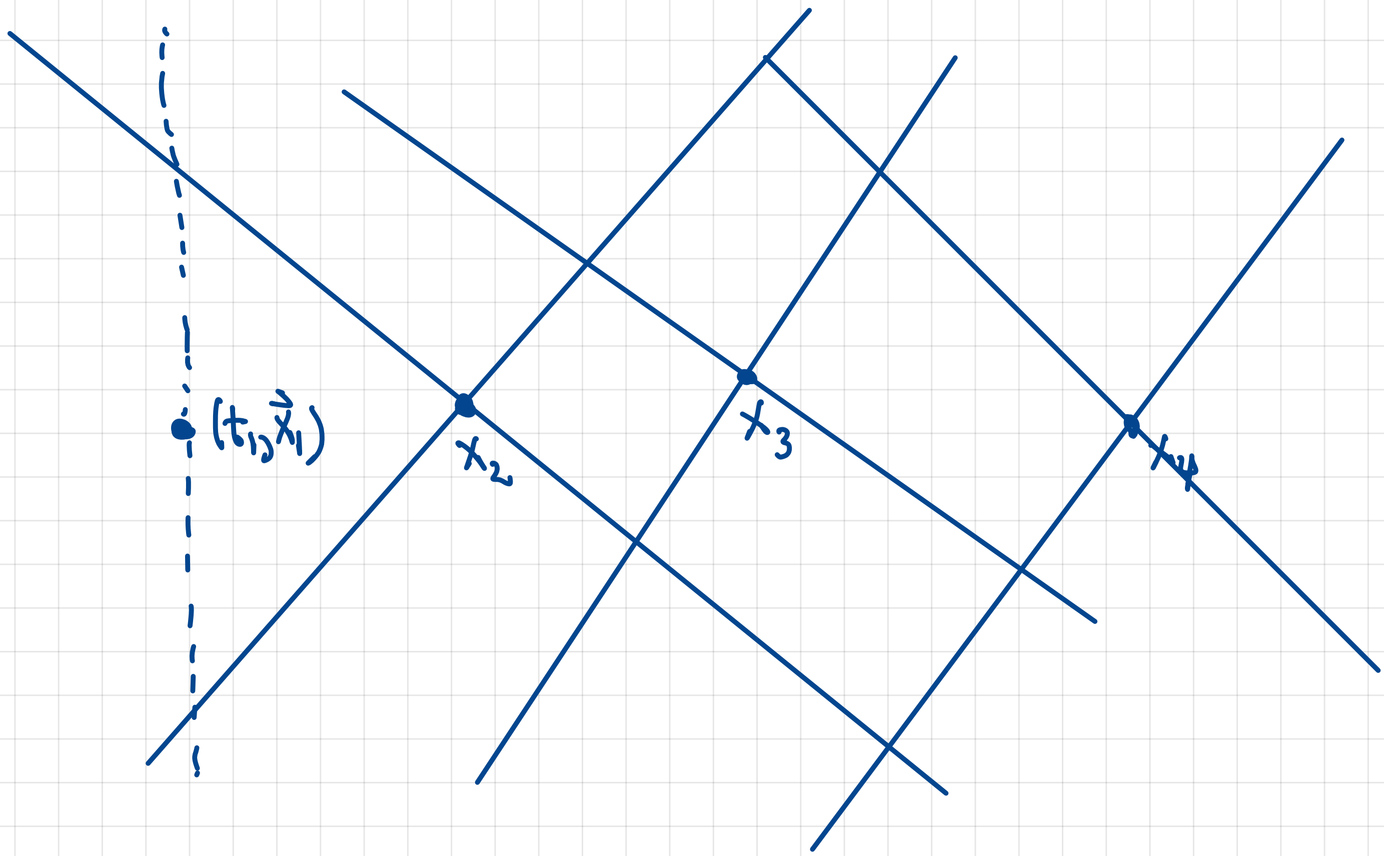
Fixes sign convention

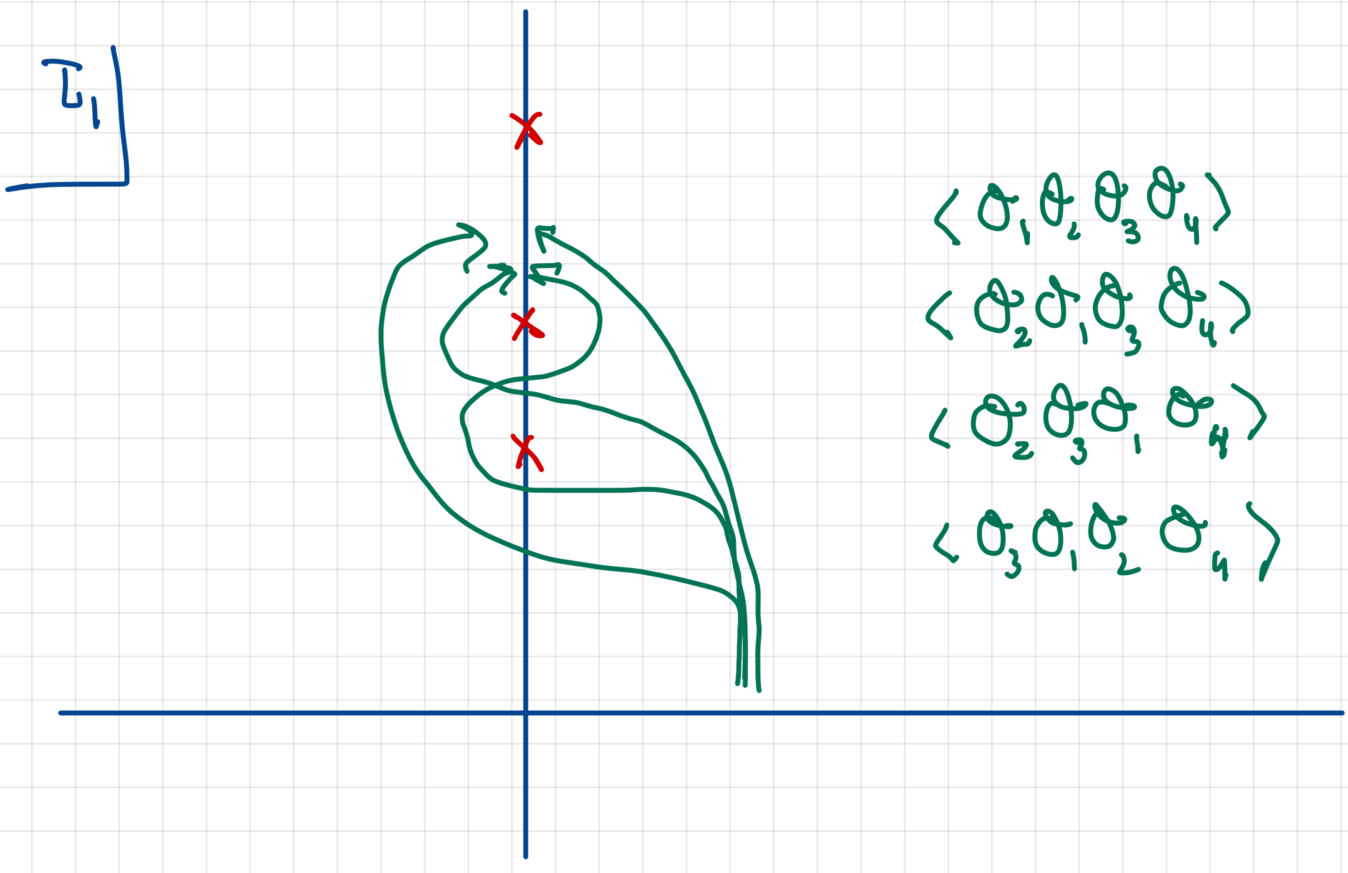
(To complete this calculation you must actually do the Fourier transform)

Note that unitarity fixed both sign of $G_{\text{Eucl.}}$ and choice of analytic continuation, for related but slightly different reasons.

ex. 4-point

$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\,\mathcal{O}(x_4) \rangle$





$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$

$\langle \mathcal{O}_2 \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_4 \rangle$

$\langle \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_1 \mathcal{O}_4 \rangle$

$\langle \mathcal{O}_3 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_4 \rangle$

$i\varepsilon$ in general

SKIP

Given Euclidean correlators $(\mathbb{R}^d)$,

$$\langle \mathcal{O}(\tau_1, \vec{x}_1)\, \mathcal{O}(\tau_2, \vec{x}_2) \cdots \rangle$$

these can be analytically continued to $\tau_i \in \mathbb{C}$ with

$$\operatorname{Re} \tau_1 > \operatorname{Re} \tau_2 > \cdots$$

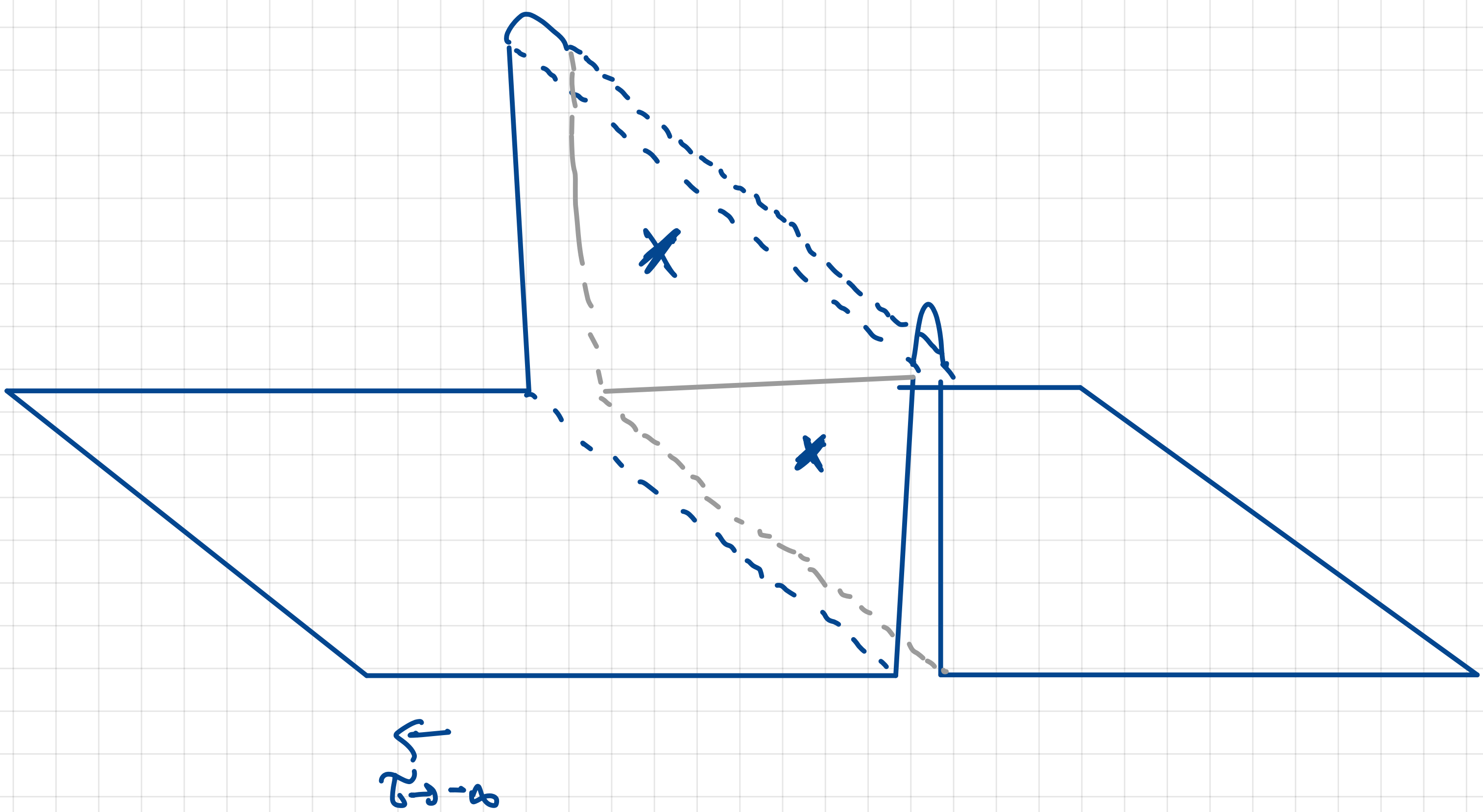
and Lorentzian correlators are

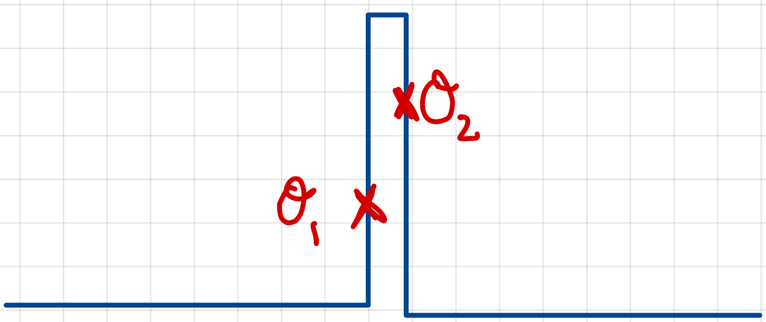
$$\langle \mathcal{O}(t_1, \vec{x}_1)\, \mathcal{O}(t_2, \vec{x}_2) \cdots \rangle$$

$$= \lim_{\varepsilon \to 0} \left\langle \mathcal{O}\left(\tau_1 = i(t_1 - i\varepsilon), \vec{x}_1\right) \mathcal{O}\left(\tau_2 = i\left(t_2 - \tfrac{1}{2} i\varepsilon\right), \vec{x}_2\right) \cdots \right\rangle$$

Path integral:

$$\langle 0 | \mathcal{O}(x_1) \mathcal{O}(x_2) | 0 \rangle =$$





vs.