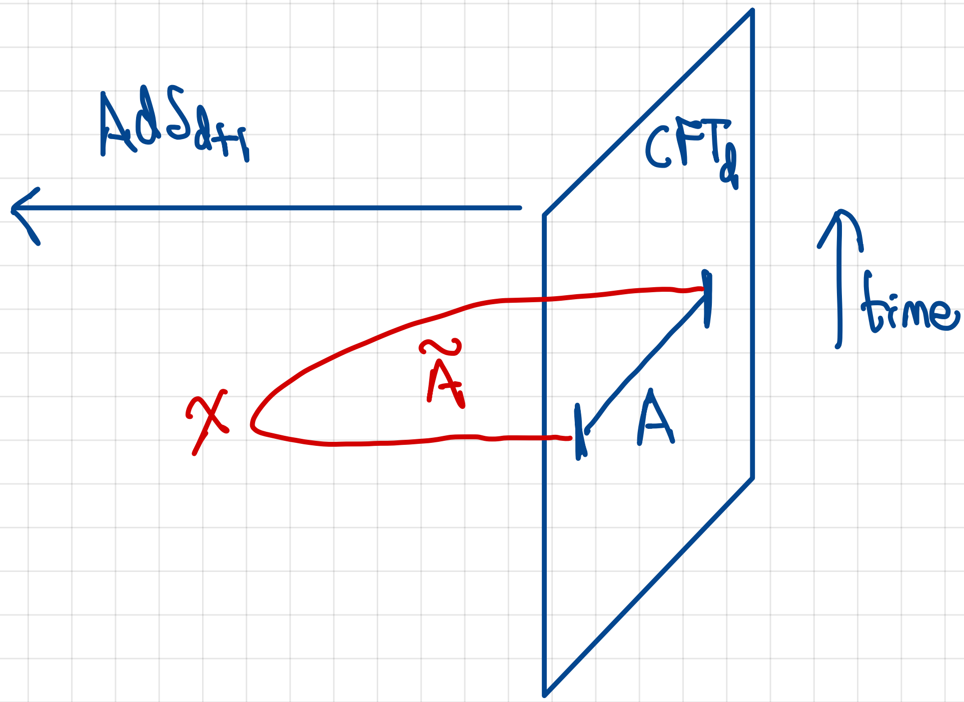
18. Holographic Entanglement Entropy

Read: QGBH § 21 and § 22.



$$\boxed{S(\rho_A) = \min_{\tilde{A}} \operatorname{ext}_{\tilde{A}} S_{gen}(\tilde{A})}$$

von Neumann

$$S_{gen}(\tilde{A}) = \frac{1}{4}\,\text{Area}(\chi) + S(\tilde{\rho}_{\tilde{A}})$$

ρ_A = density matrix of dual CFT OR QG.

$\tilde{\rho}_{\tilde{A}}$ = density matrix of bulk QFT

$$\text{"}\min_{\tilde{A}} \underbrace{\text{ext}_{\tilde{A}}}\text{"}$$

local min. under spacelike deformations
" max. " timelike "

If multiple local extrema, pick global min S_{gen}.

min/ext over spacelike $\tilde{A}$ satisfying "homology condition:"

$$\partial \tilde{A} = A \cup \chi$$

at extremum,

χ = "Quantum extremal surface"

$\tilde{A}$ = "Entanglement Wedge"

Ryu-Takayanagi / HRT formula

Often S_{QFT} is subleading, so

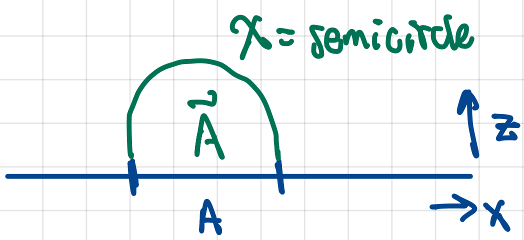
$$S(\rho_A) \approx \min_{\tilde{A}} \operatorname{ext} \frac{1}{4} \text{Area}(\chi)$$ "HRT"

In static spacetime,

$$S(\rho_A) = \min_{\tilde{A}} \frac{1}{4} \text{Area}(\chi)$$ "RT"

Examples

2d CFT in Vacuum



RT formula:

$$S(\rho_A) = \frac{1}{4G} \text{Length}(\chi)$$

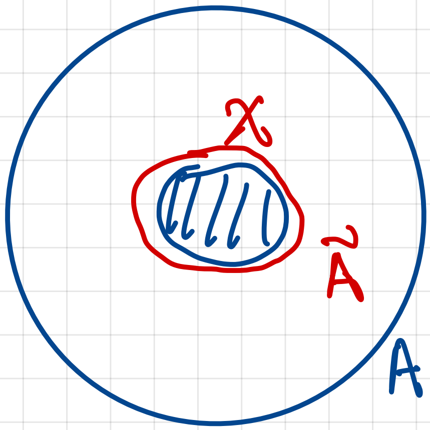
$$= \infty \quad \ldots$$

Cutoff @ $z > \varepsilon$ $\Rightarrow$

$$S(\rho_A) = \frac{c}{3} \log\left(\frac{\ell(A)}{\varepsilon}\right) \quad \checkmark$$

used: $c = \dfrac{3\ell_{AdS}}{2G}$

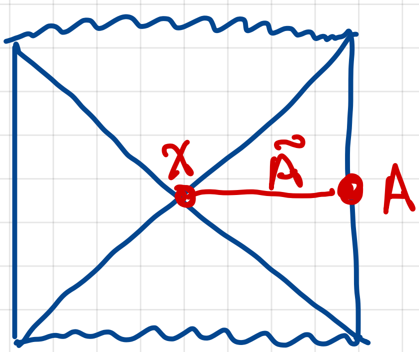
Eternal Black Hole



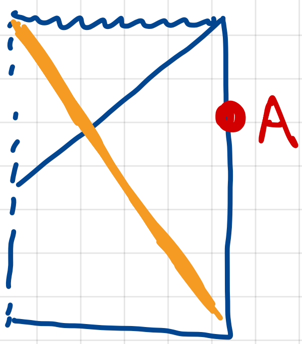
A = everything

$$S(\rho_A) = \frac{1}{4}\,\text{Area}(\chi)$$

$$= \text{Black hole entropy} \quad \checkmark$$



Black Hole formed from pure state

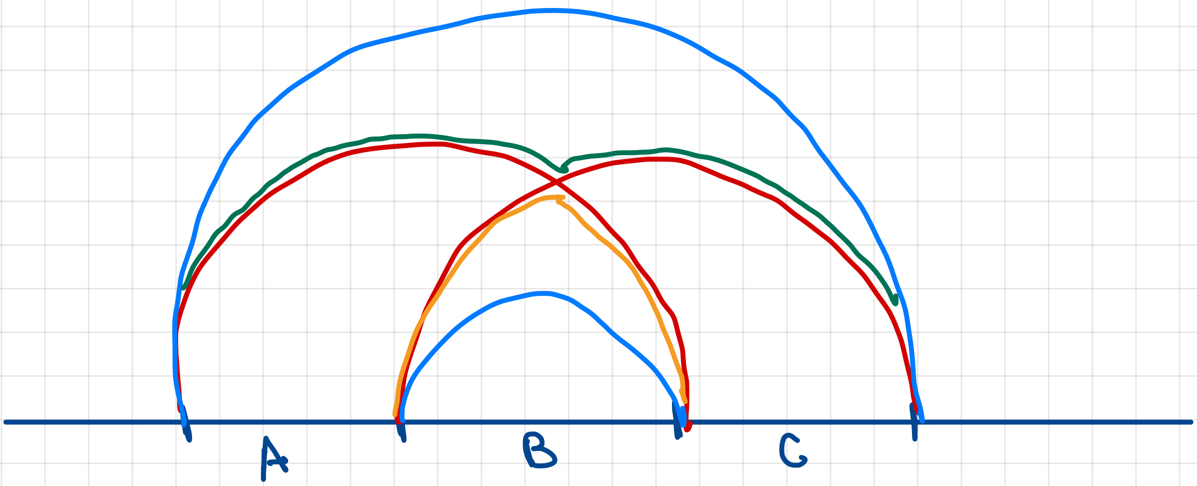


$\tilde{A}$ = nothing

$$S(\rho_A) = 0 \quad \checkmark$$

SSA

[Skip in lecture]



$$\underbrace{S_{AB} + S_{AC}}_{} \geq S_B + S_{ABC}$$

red + red = yellow $\geq$ small blue = S_B

+green $\geq$ big blue = S_{ABC}

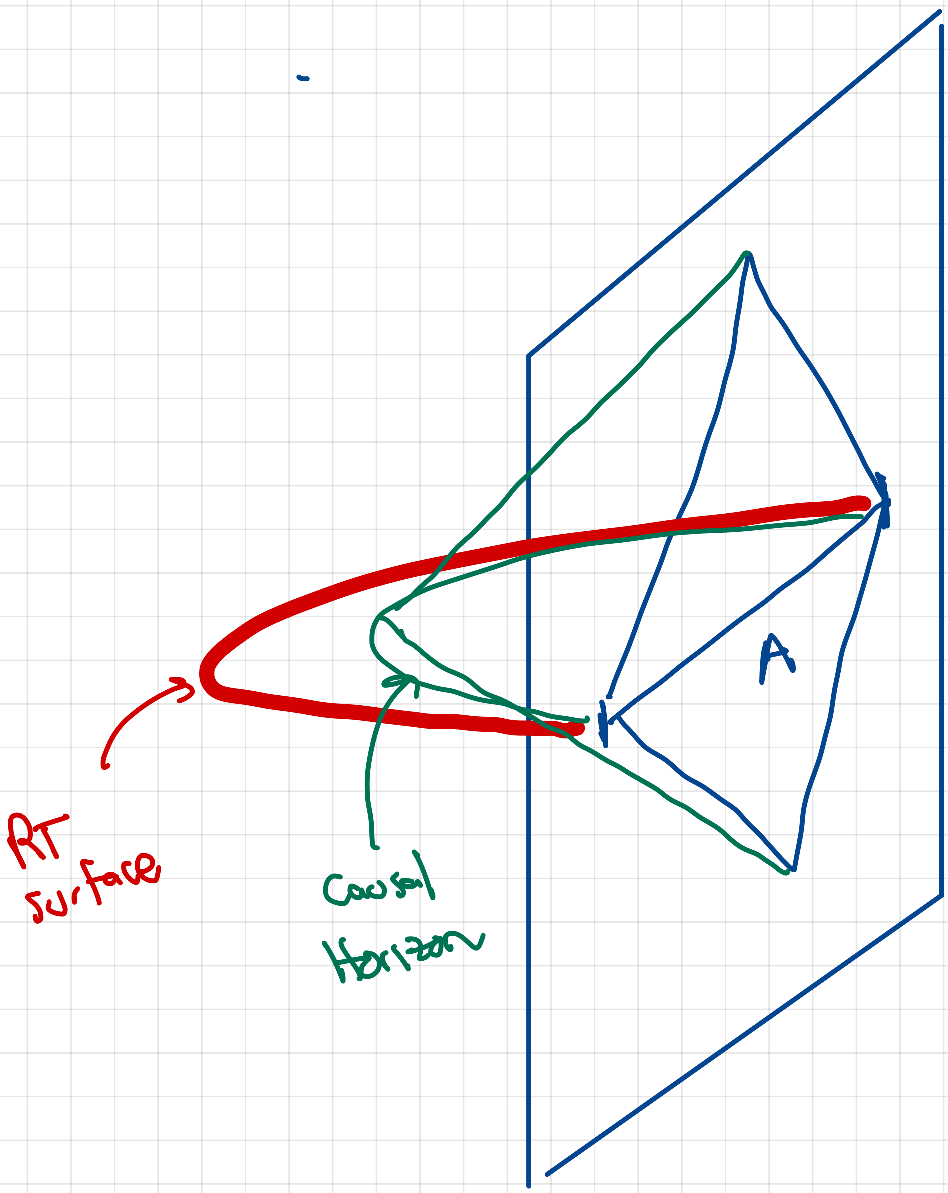
$\checkmark$

Bulk Reconstruction

Claim: Bulk region $\tilde{A}$ is "encoded" in ρ_A

(Go through the examples again to explain...)

Causal vs. Entang. Wedge:

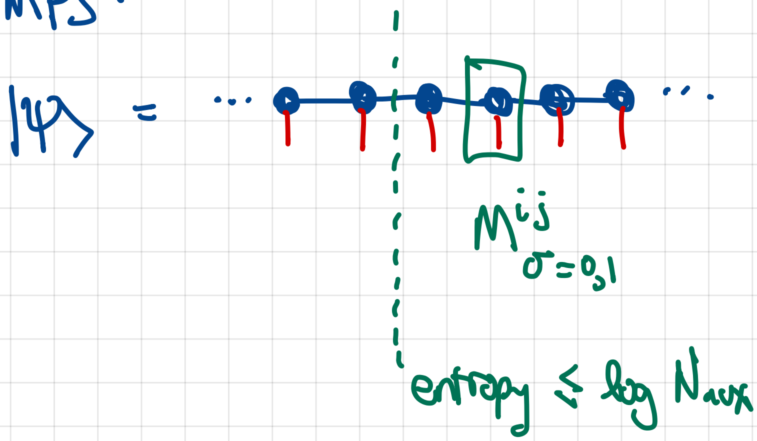


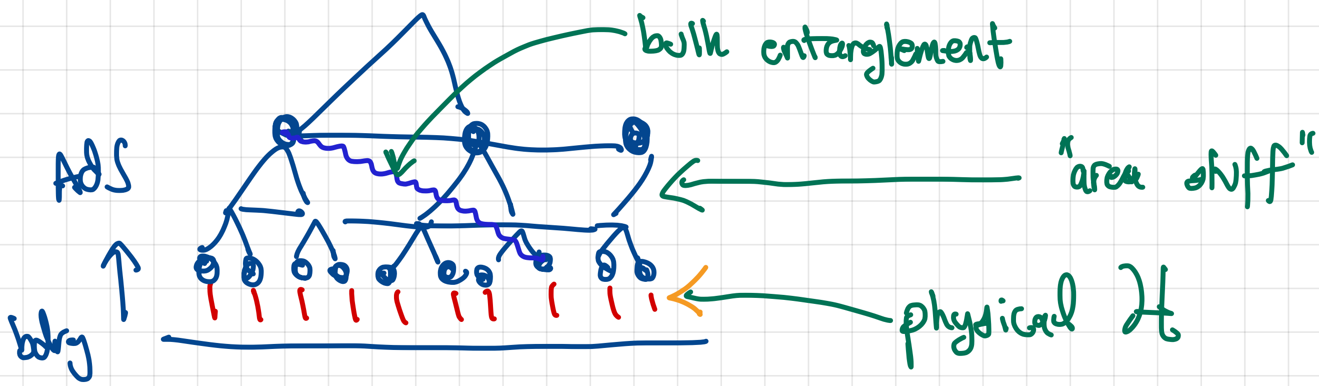
Intuition

Explain via shockwaves

Tensor network toy model

Recall MPS:

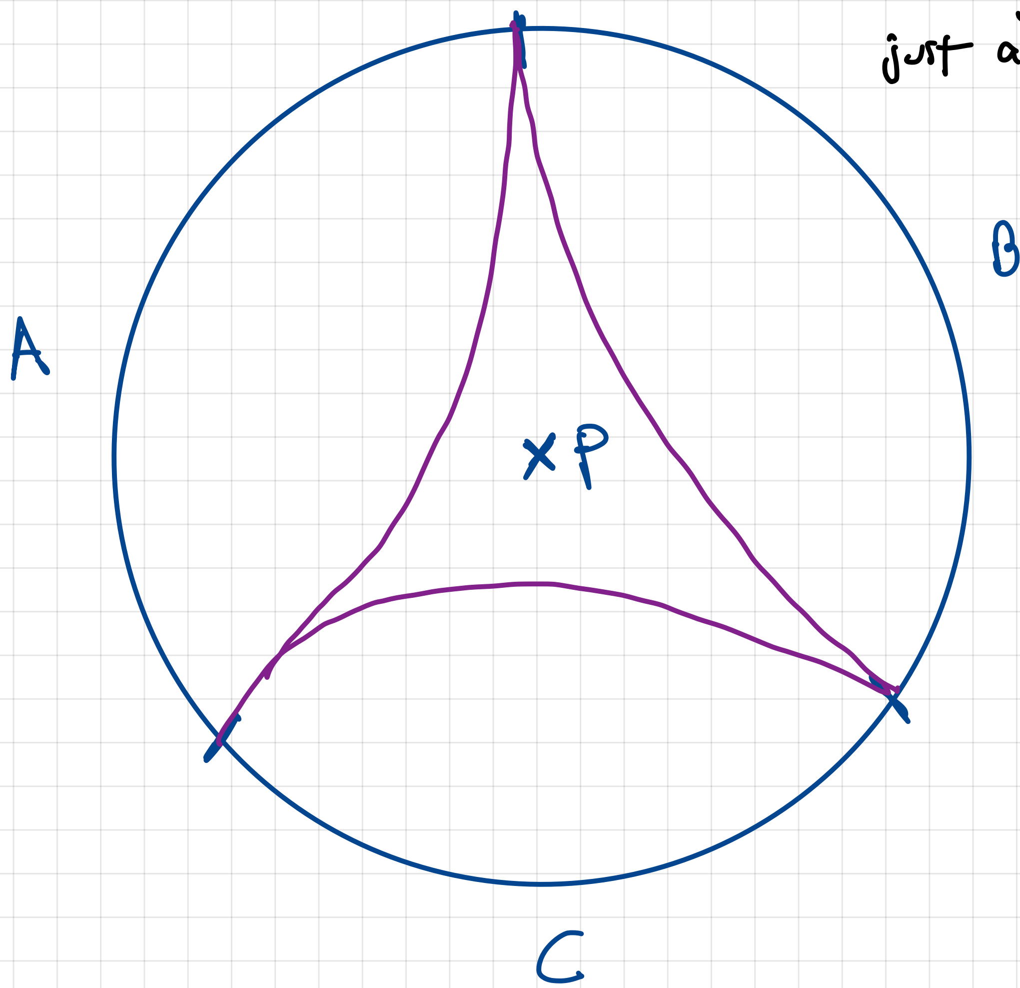




(Explain area law: random tensors?)

"Quantum Error Correcting Code"

Same idea, but not discretized, so this is actually accurate - not just a toy model!



"p" is not encoded in ρ_A, ρ_B, or ρ_C,

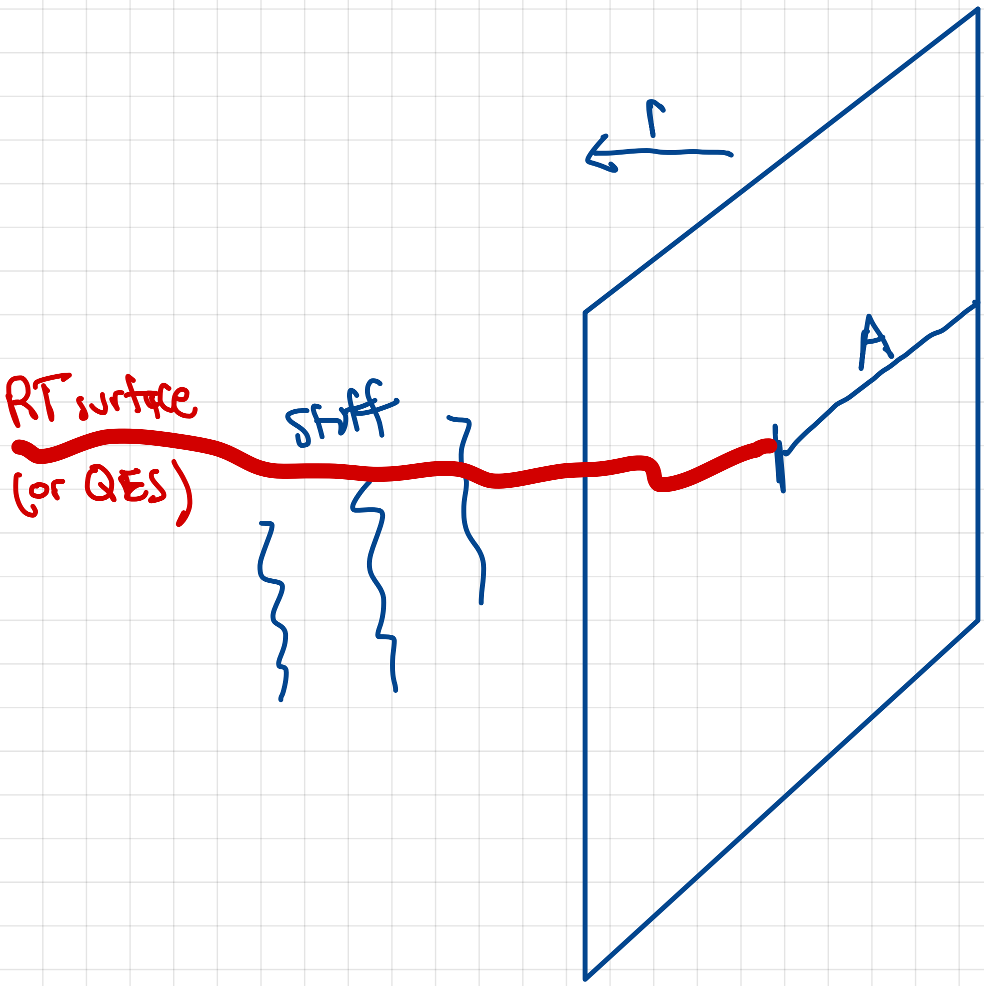
but is encoded (redundantly) in

ρ_{AB} OR ρ_{AC} OR ρ_{BC}

Very Sketchy Derivation of HEE formula

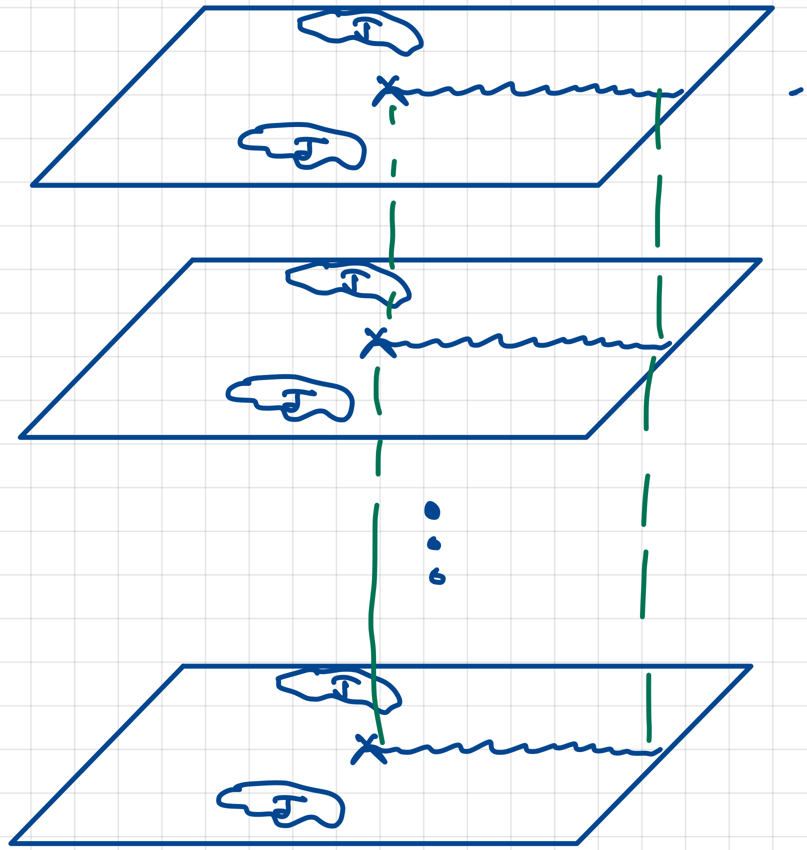
We'll do A = half-space for pictorial purposes, but the discussion is general (and any D.)

Lorentzian Setup:

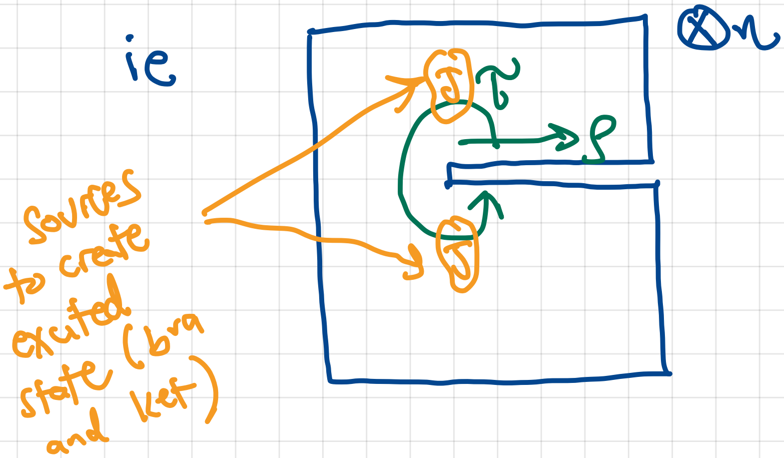


Calculate $\mathrm{Tr}\,\rho_A^n$ by Euclidean P.I.

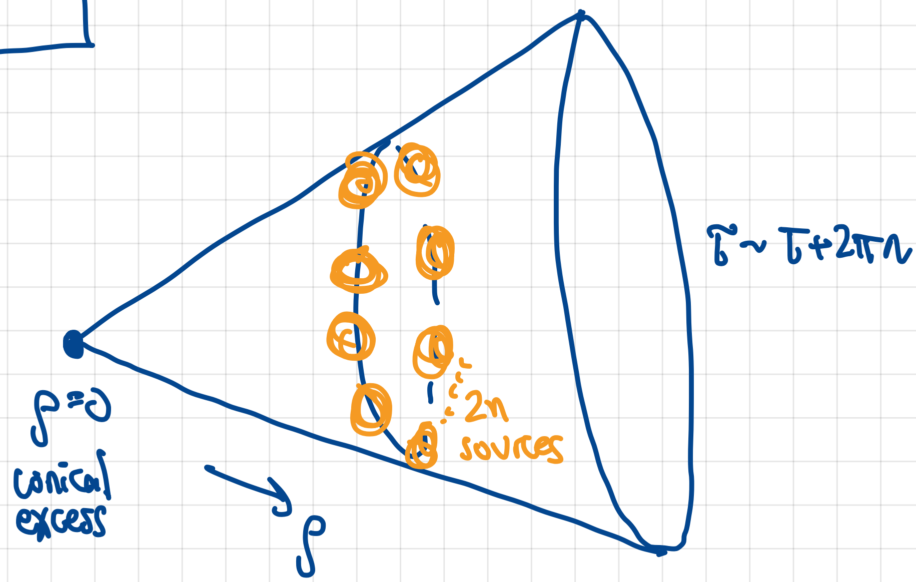
$\mathrm{Tr}\,\rho_A^n$ = gravity path integral with boundary condition



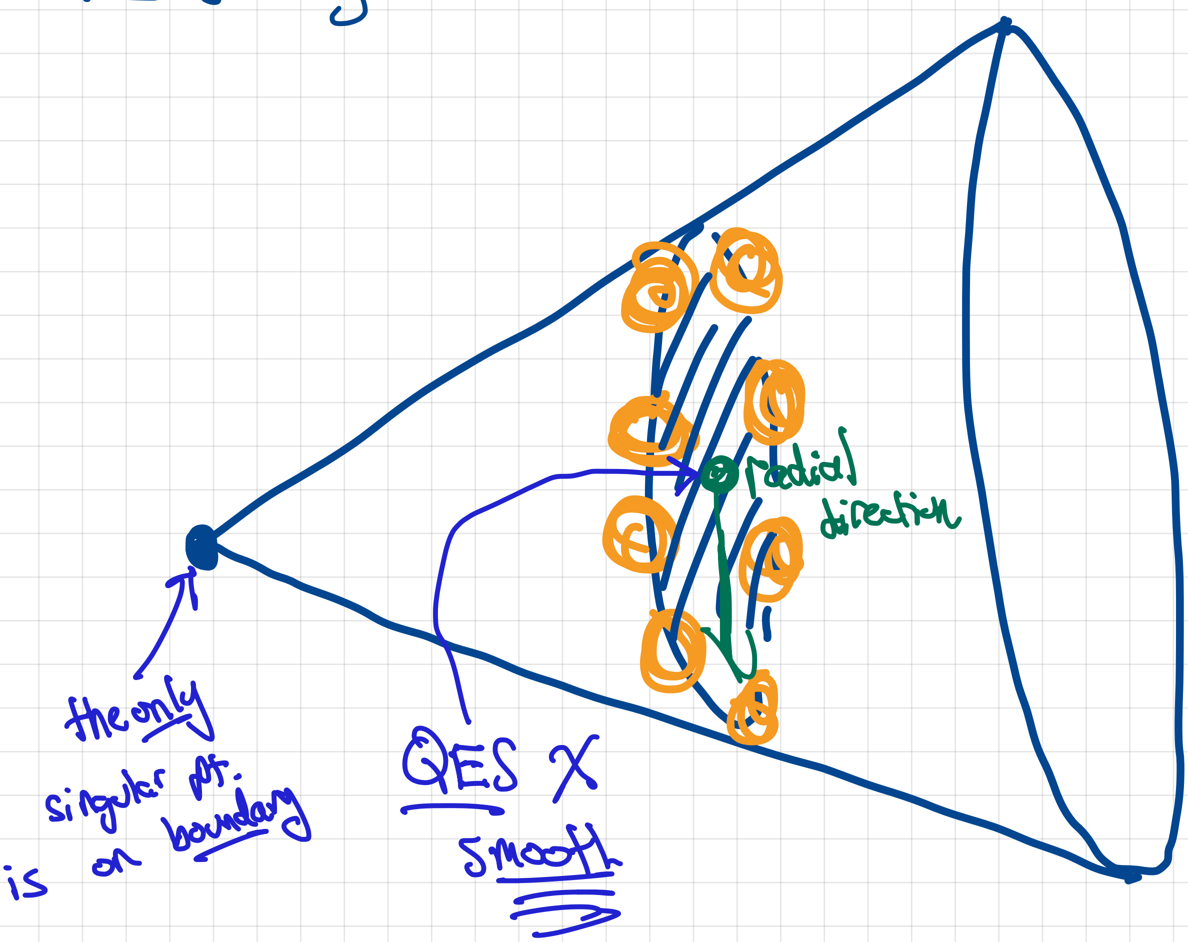
ie



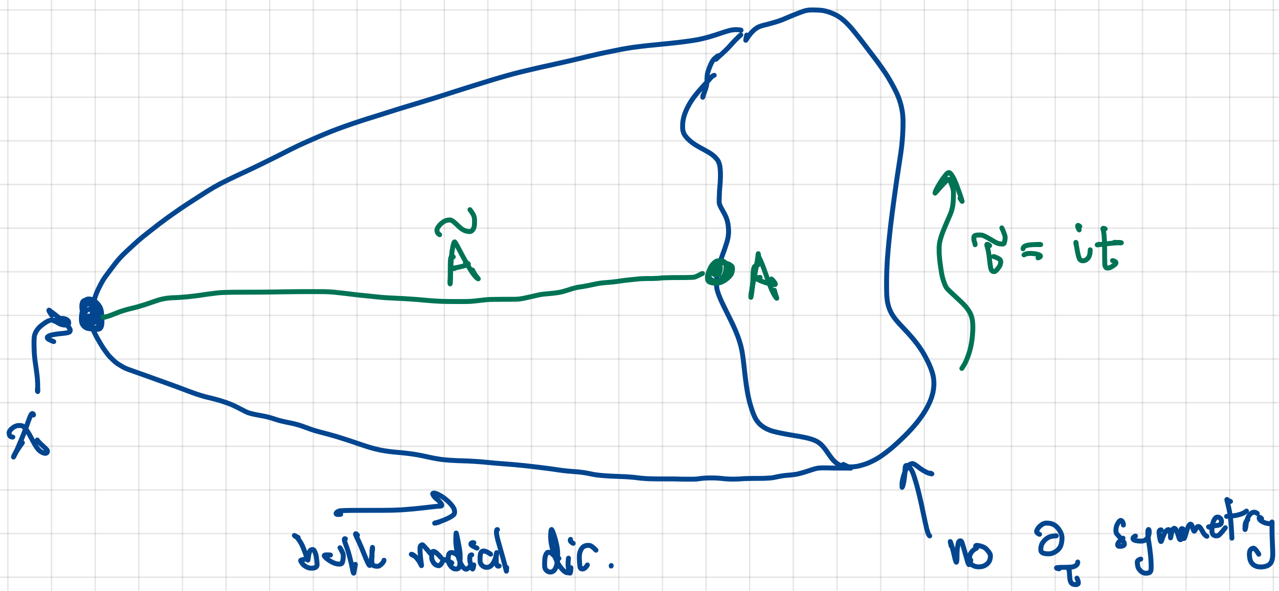
= "cone"



<u>Bulk</u> geometry = "filled cone" (topologically)



Often this is drawn at fixed ρ :



Skip in lecture

For $n \sim 1$, and fixed x,

$$e^{-S_{grav}} Z_{QFT} \sim e^{(1-n)\left[\frac{1}{4}\mathrm{Area}(x) + S_{QFT}\right]}$$

off-shell effective action

$$\Rightarrow \quad \frac{Z(n)}{Z(1)^n} \sim \int d\begin{pmatrix}\text{choice of} \\ x\end{pmatrix} \exp\left[(1-n)\, S_{gen}(\tilde{A})\right]$$

$$\sim \exp\left[(1-n)\, S_{gen}(\tilde{A}^*)\right]$$

$$S(\rho_A) = \frac{1}{1-n} \log \operatorname{tr} \rho_A^n = S_{gen}(\tilde{A}^*) \quad \checkmark$$

CAVEAT

Derivation is not microscopic, so

$$S_{replica}(\rho_A) = -\operatorname{tr} \rho_A \log \rho_A$$

must be viewed as conjecture.