17.

Generalized Entropy

Definition

$$S_{gen} = \frac{1}{4}\,\text{area}(BH) + S_{out}$$

S_{out} = von Neumann entropy of QFT outside, (including gravitons)

$$= S(\rho_{out}^{QFT})$$

<u>OR</u>, sometimes

S_{out} = "coarse-grained" $S(\rho_{out}^{QFT})$

S_{gen} is UV-finite @ Horizon

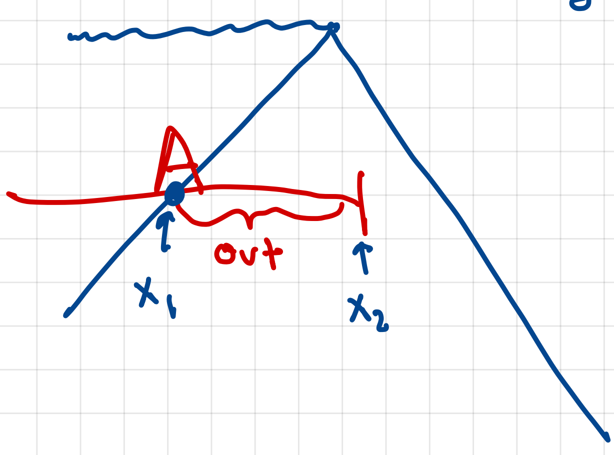
Roughly,

$$S_{gen} = \frac{\text{Area}}{4G_N^{0}} + \underbrace{S_{out}}_{\frac{\text{Area}}{\varepsilon^{d-2}} + \cdots + \text{finite QFT part}}$$

G_N^0: bare

$$= \frac{\text{Area}}{\underbrace{4G_N}_{\text{renormalized, finite}}} + \text{finite QFT part}$$

example: 2d



$$S_{gen}(A) \equiv \frac{\text{area}(x_1)}{4G_N^0} + \frac{c}{6}\log\left(\frac{(x_2-x_1)^2}{\underbrace{\varepsilon_1\,\varepsilon_2}_{\text{UV cutoffs}}}\right)$$

$$= \frac{\text{area}(x_1)}{4G_N} + \frac{c}{6}\log\left(\frac{(x_1-x_2)^2}{\underbrace{\mu}_{\text{RG scale}}\;\underbrace{\varepsilon_2}_{\text{UV divergence b/c no area term.}}}\right)$$

Finiteness has been verified in some cases (free scalars, etc.) but generally we have only a formal argument, coming below.

Generalized 2nd Law (GSL)

(conjecture)

$$S_{gen}(\text{universe}) = \sum_{\text{all BH}} \frac{\text{area}}{4} + S_{out}$$

increases in all physical processes.

$$\frac{d}{dt} S_{gen} \geq 0$$

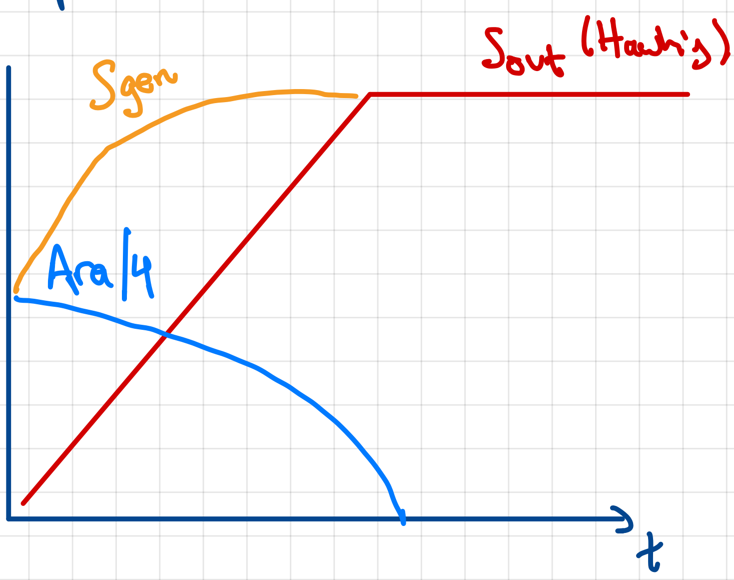
Subtlety

Supposed to hold for S_{out} computed in perturbative QFT, or with coarse-grained S_{out}.

NOT w/ exact $S(\rho_{out})$.

Ex.

BH evaporation



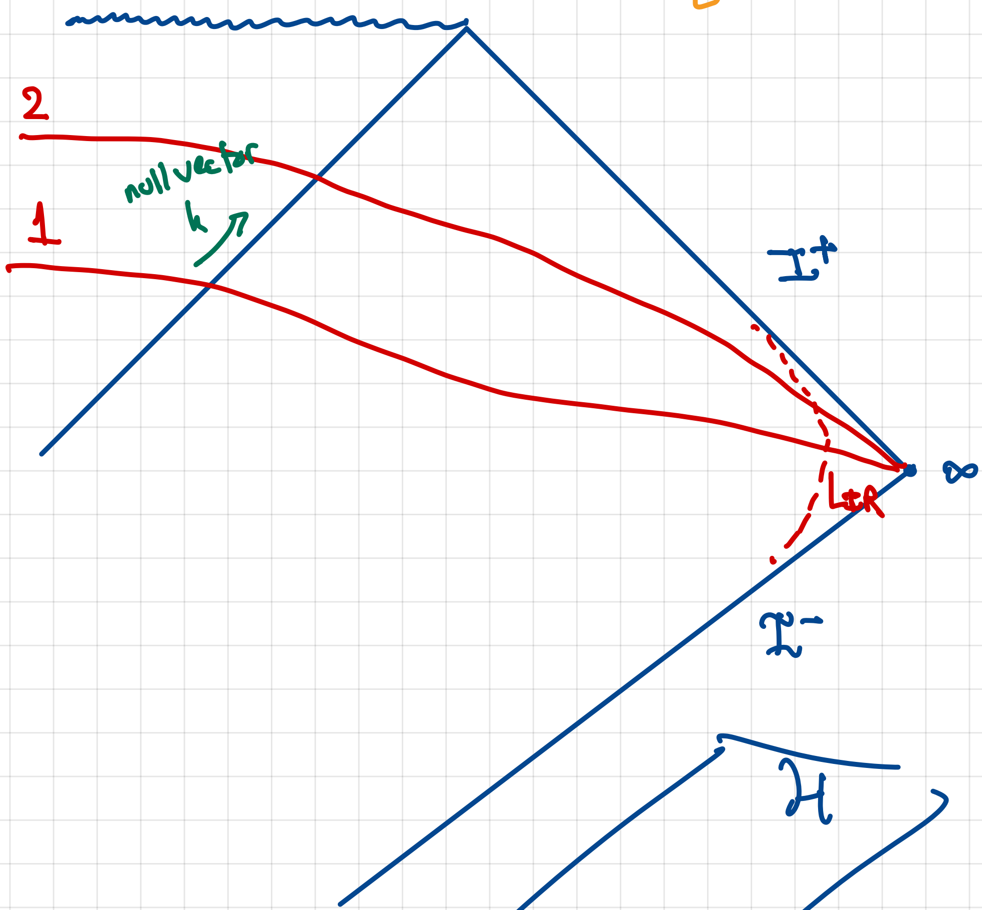
Note something tricky happened here. 2nd law applies to coarse-grained entropies.

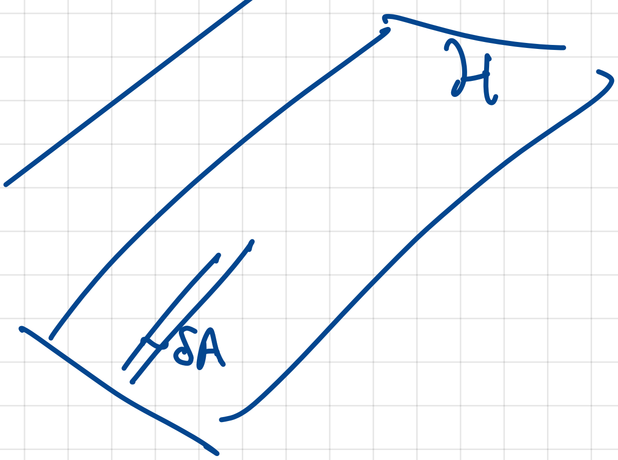
Here we are applying to fine-grained entropy of perturbative QFT - and it works!!

Does perturbative QFT secretly compute a coarse-grained entropy?

Wall's Proof of the GSL

[Skip in lecture – use short version below]





Area Term

$$\theta = \frac{1}{\delta A}\frac{d}{d\lambda}\delta A$$

$$\nabla_k \theta = -\frac{\theta^2}{D-2} - \sigma_{ab}\sigma^{ab} - \underbrace{R_{kk}}_{=8\pi G_N T_{kk}^{\text{matter}}}$$

The classical area theorem follows from NEC + no singularities on $\mathcal{H}$.

Assume horizon is classically stationary (otherwise GSL is trivial)

So $\Theta^2 \sim \hbar^2 \Rightarrow$

$$\nabla_k \Theta = \langle -\sigma_{ab}\sigma^{ab}\rangle - 8\pi G_N \langle T^{\text{matter}}_{kk}\rangle$$

$$\nabla_k \Theta = -8\pi G_N \langle T_{kk}\rangle$$

$$\nabla_k^2 \delta A = -8\pi G_N \langle T_{kk}\rangle \delta A$$

(extra terms $\nabla A \nabla A \sim \hbar^2$)

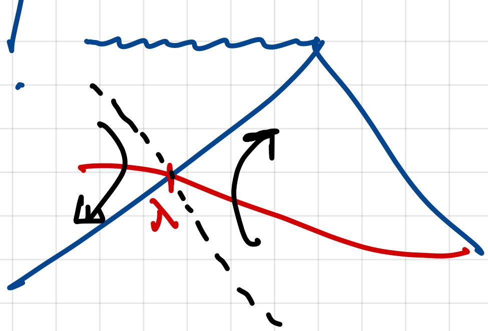
Integrate twice

$$\delta A(\infty) - \delta A(\lambda) = 8\pi G_N \int_\lambda^\infty d\lambda' \, (\lambda' - \lambda) \langle T_{kk}(\lambda')\rangle$$

$$A(\lambda) = \text{const.} - 8\pi G_N \langle K(\lambda)\rangle$$

$$\langle K(\lambda)\rangle \equiv \int dy_\perp \int_\lambda^\infty d\lambda' \, (\lambda' - \lambda) \langle T_{kk}(y_\perp, \lambda')\rangle$$

$= \text{Boost Charge}$!



Let

$$\sigma_{out} = e^{-2\pi Q_{boost}} \qquad \left(\text{thermal @ } T = \frac{1}{2\pi}\right)$$

Thus

$$\frac{d}{d\lambda} A(\lambda) = - 8\pi G_N \left(\frac{-1}{2\pi}\right) \frac{d}{d\lambda} \log \sigma_{out}$$

$$\frac{d}{d\lambda} \left(\frac{A(\lambda)}{4G_N}\right) = \frac{d}{d\lambda} \log \sigma_{out}$$

Matter: Relative Entropy

ρ_{out} = actual state

σ_{out} = thermal

$$S(\rho|\sigma) = \operatorname{tr} \rho \log \rho - \operatorname{tr} \rho \log \sigma$$

$$= - S(\rho) - \langle \log \sigma \rangle_\rho$$

$$\frac{d}{d\lambda} S(\rho|\sigma) \leq 0 \qquad \text{monotonicity of rel. ent.}$$

"shrinks region"

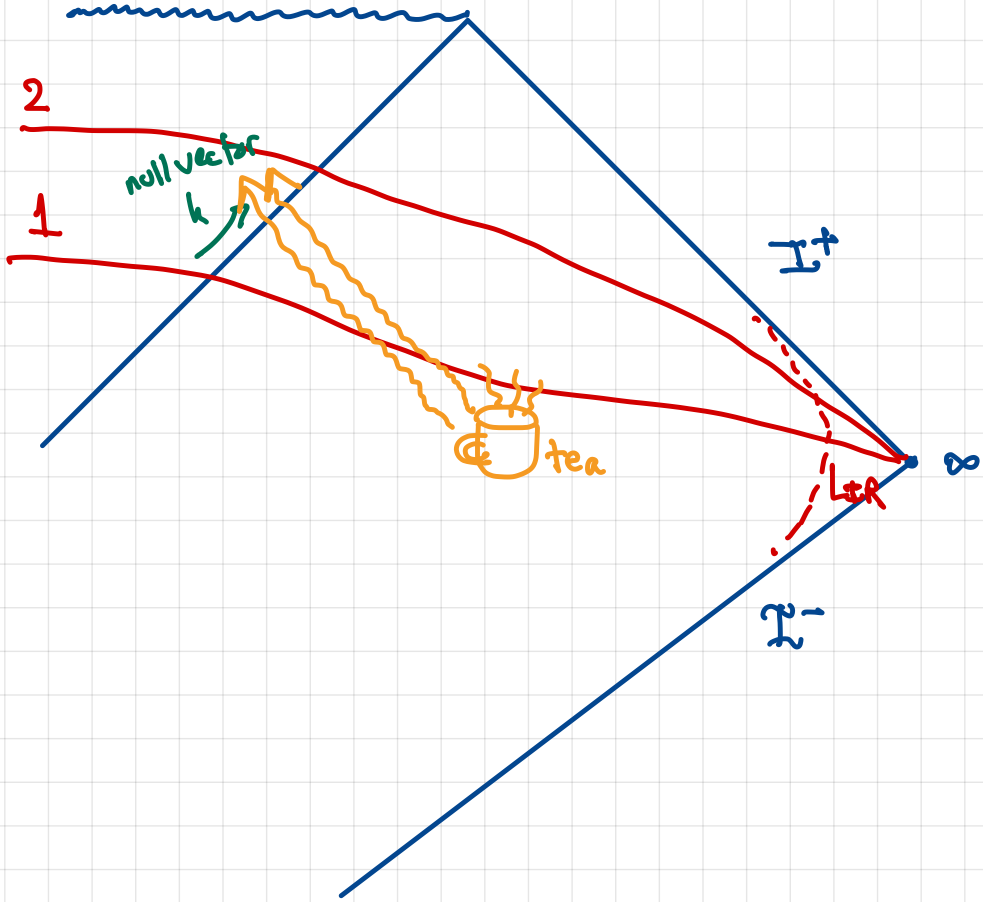
$$\Rightarrow \quad \frac{d}{d\lambda}\left[-S(\rho_{out}) - \langle \log \sigma \rangle_{\rho} \right] \leq 0$$

$$\Rightarrow \quad \boxed{\frac{d}{d\lambda}\left[S(\rho_{out}) + \frac{\text{Area}}{4G_N} \right] \geq 0}$$

GSL

Alt.

Short Version of Wall's Proof - SKETCH



Raychaudhuri $\Rightarrow$

$$A(\lambda) = \text{const.} - 8\pi G_N \underbrace{\int d^{d-2}y_\perp \int_\lambda^\infty d\lambda' \, (\lambda' - \lambda) \, \langle T_{kk}(\lambda', y_\perp) \rangle}_{\text{Boost charge } \langle K(\lambda) \rangle}$$

[draw boost]

[describe flux response]

$$\frac{dA(\lambda)}{d\lambda} = -8\pi G_N \frac{d}{d\lambda} \langle K(\lambda) \rangle$$

Matter Relative Entropy

$$S(\rho|\sigma) = \operatorname{tr}\rho\log\rho - \operatorname{tr}\rho\log\sigma$$
$$= -S(\rho) - \langle \log\sigma\rangle_\rho$$

ref. state:

Let σ = HH state

$$\sigma_{out} = \exp\left(-2\pi\langle K(\lambda)\rangle\right)$$

thermal @ temp. $1/2\pi$

[Not the actual state; we're dropping stuff in!]

$$S(\rho_{out}|\sigma_{out}) = -S(\rho_{out}) + 2\pi\langle K(\lambda)\rangle$$
$$= -S(\rho_{out}) - \frac{1}{4G_N}\text{Area} + \text{const.}$$

Monotonicity of S_{rel}:

$$\frac{d}{d\lambda} S(\rho_{out}|\sigma_{out}) \leq 0$$

"shrinks" region 'out' $\Rightarrow$ less distinguishable

$$\Rightarrow \boxed{\frac{d}{d\lambda}\left[\frac{\text{Area}}{4G_N} + S(\rho_{out})\right] \geq 0}$$

GSL = MRE!

Replica Derivation of S_{gen}

(Following Lewkowycz-Maldacena, 1304.4926)

Claim: for a stationary BH,

$$S(\rho) = S_{gen} = \frac{1}{4}\text{Area} + S(\rho^{QFT})$$

vN in quantum gravity

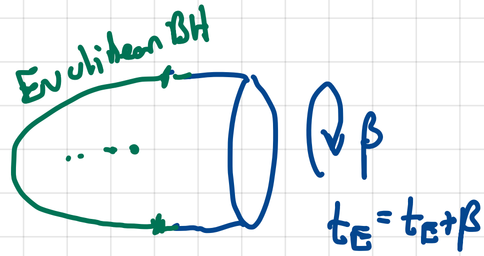
vN in QFT

Replica Method

$$\mathrm{Tr}\,\rho^n = \mathrm{Tr}\,e^{-n\beta H}$$

recall $n=1$, classical approx:

$$\mathrm{Tr}\,e^{-\beta H} = $$



$$\approx e^{-S_E(\text{euclidean BH})}$$

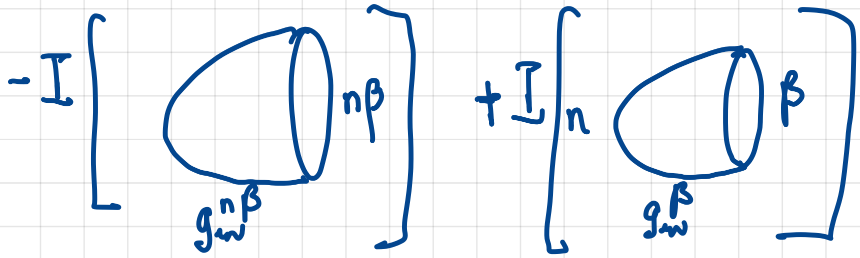
now $n \neq 1$

$$Z(n) = \mathrm{Tr}\,e^{-n\beta H}$$

$$S = -n\partial_n \log\left(\frac{Z(n)}{Z(1)^n}\right)\Bigg|_{n=1}$$

$$= -n\partial_n\Big[\log Z(n) - n\log Z(1)\Big]_{n=1}$$

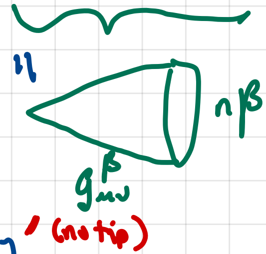
brackets =

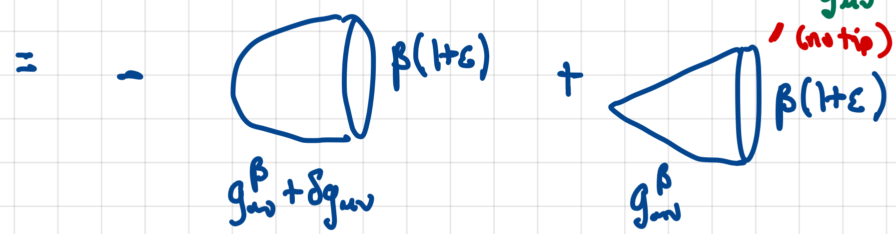


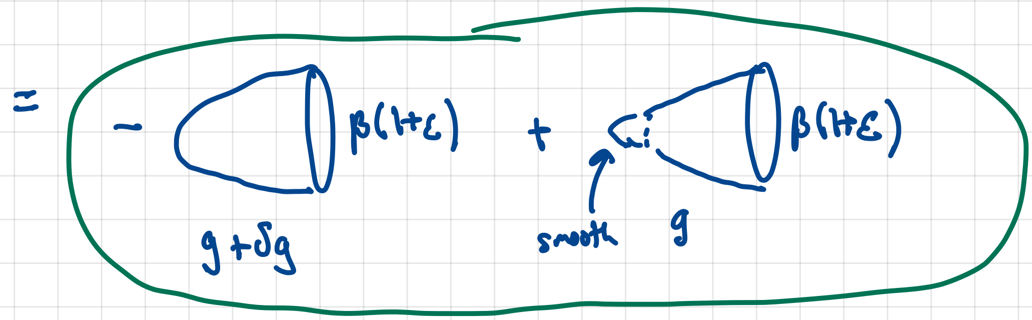
All Smooth b/c of grav. backreaction (this is why UV-finite)

(compare QFT entropy)

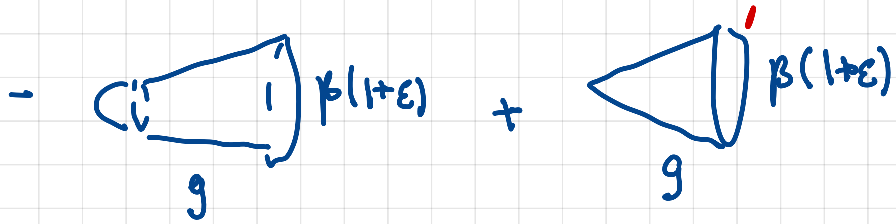
$n = 1+\varepsilon$





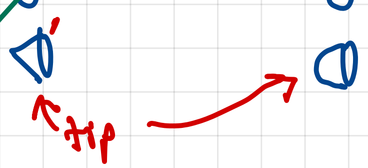


0 because $\delta I_{\text{on-shell}} = 0$



0 b/c not including any singular part from tip.

$$= \frac{-1}{16\pi G}\int \sqrt{g}\, R \qquad + \frac{1}{16\pi G}\int \sqrt{g}\, R$$



$$= \frac{1}{16\pi G}\int \sqrt{g}\, R \quad = \quad \frac{4\pi}{16\pi G}(1-n) \times \left(\text{Area in transverse dirs} \neq t_E, r\right)$$

Last equality = Gauss-Bonnet theorem,

$$\int_{2d\,Mf} \sqrt{g}\,R = 4\pi\,\chi(M) - \int_{\partial M} ds\,k$$

$$\chi(\text{Disk}) = 1$$

$$k_{\text{circle}} = \frac{2}{R}$$

$$= 4\pi - \frac{2}{R}\int ds$$

$$= 4\pi - \frac{2}{R}(2\pi n R)$$

$$= 4\pi(1-n)$$

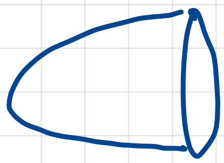
$$S = -n\partial_n \left[\frac{1}{4} A(1-n) \right]_{n=1}$$

$$\boxed{S = \frac{1}{4}\,\text{Area} \quad \checkmark}$$

(See Faulkner, Lewkowycz, Maldacena 1307.2892)

Quantum Corrections

$Z(n) =$



$$\approx e^{-I_{grav}(n\beta)} Z_{QFT}(n\beta)$$

$$\Longrightarrow \quad \ldots$$

backreaction of $T^{QFT}_{\mu\nu}$ on Euclidean BH.

$$S = \frac{\text{Area}^{(0)}}{4} + \frac{\delta\text{Area}}{4} + S\left(\rho^{QFT}_{out}\right)$$

vN entropy of QFT on

$$= S_{gen} \quad \checkmark$$