16.

Replica Method in CFT

Based on the paper hep-th/0405152 by Cardy, Calabrese

Goal:

Consider a 2d CFT on $\mathbb{R}$:

In vacuum, $\rho = |0\rangle\langle 0|$, find $S(\rho_A)$

Replica Method:

$$S(\rho) = -\mathrm{tr}\, \hat{\rho} \log \hat{\rho}$$

hard

$$Z(n) = \mathrm{tr}\rho^n$$

$\hat{\rho}$ = normalized
ρ = not. , so $\hat{\rho} = \dfrac{\rho}{\mathrm{tr}\rho}$

$$\mathrm{tr}\hat{\rho}^n = \frac{Z(n)}{Z(1)^n}$$

$$\hat{\rho}^{1+\varepsilon} = \hat{\rho} + \varepsilon\, \hat{\rho} \log \hat{\rho} + \dots$$

$\therefore$

$$S(\rho) = -\partial_n \left(\frac{Z(n)}{Z(1)^n} \right)_{n=1}$$

Equivalently,

"Renyi entropy" $S_n = \frac{1}{1-n} \log \operatorname{tr} \hat{\rho}^n$

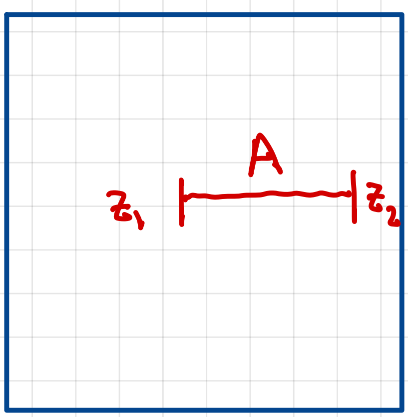
$$S(\rho) = \lim_{n\to 1} S_n$$

Calculation of $\operatorname{tr}\rho^n$ for $n \in \mathbb{Z}$

$A = [z_1, z_2] \subset \mathbb{R}$

Recall Euclidean P.I.

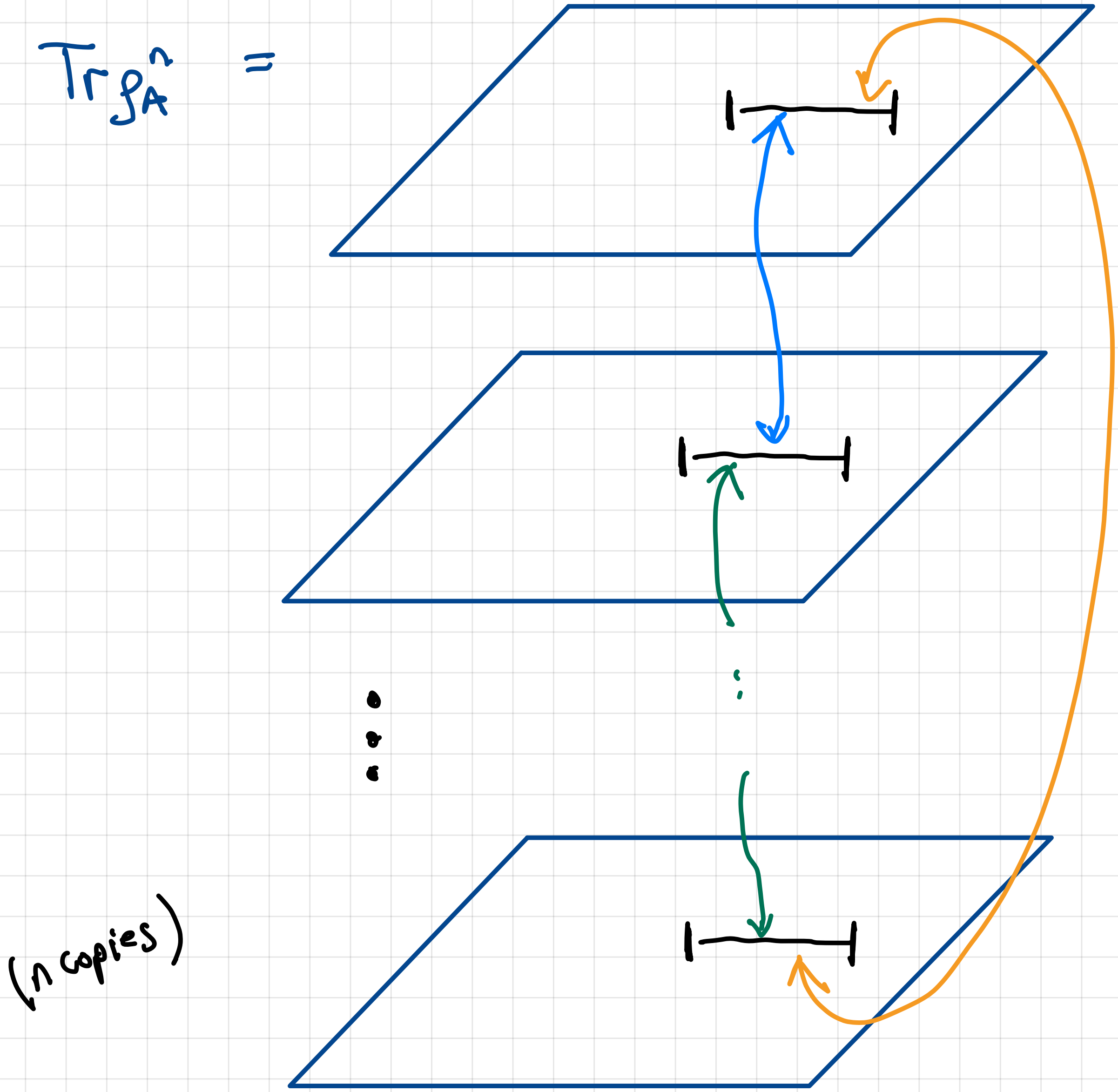
$\rho_A =$



(reminder?)

Therefore,

$\mathrm{Tr}\,\rho_A^n =$



$$= \int D\phi^{(1)}(z^{(1)})\, D\phi^{(2)}(z^{(2)}) \cdots D\phi^{(n)}(z^{(n)})\; e^{-\sum_n S_{CFT}[\phi^{(n)}]}$$

w/ gluing conditions

$$\phi^{(k)}(z^{(k)} \in A^-) = \phi^{(k+1)}(z^{(k+1)} \in A^+)$$

Let's compute this crazy path integral!

One way to proceed:

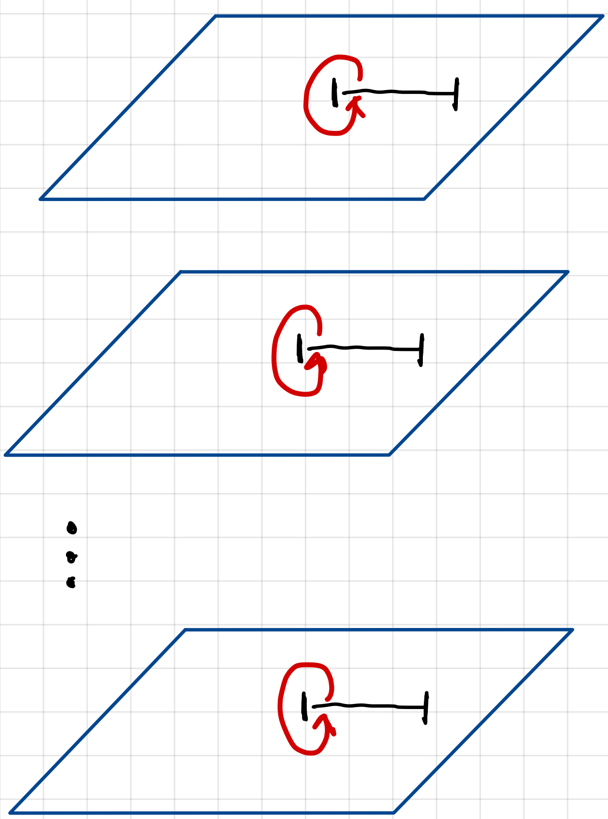
$$Z[g] = Z[\delta] \exp\left(-S_p(g)\right)$$

$g_{\mu\nu}$ on replica mf. is $= \delta_{\mu\nu}$

except at $z = z_1, z_2$

These points are conical excesses:

Small circle of radius R has length $2\pi R n$



This is tricky, but doable.

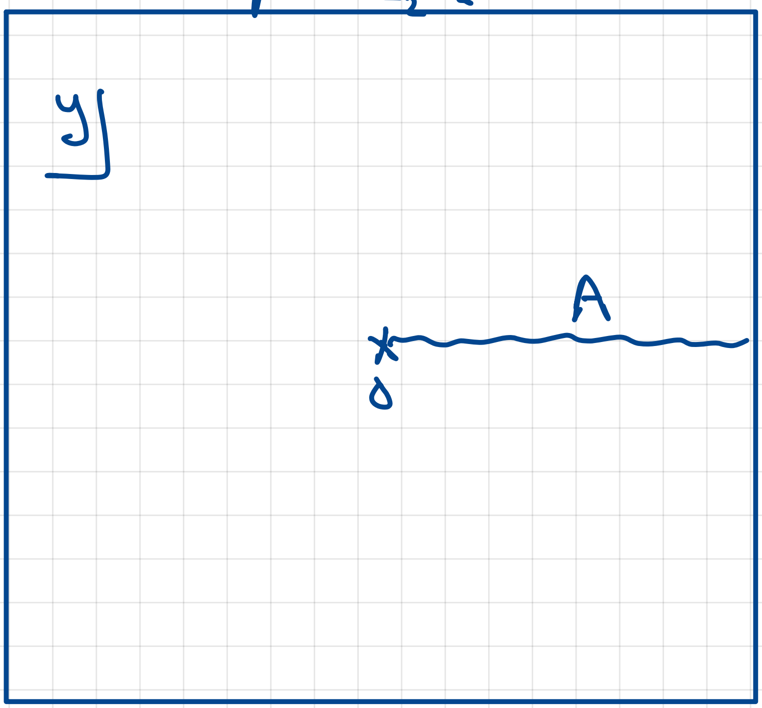
Instead:

"Uniformize"

Any topologically trivial 2-mf. without boundaries can be conformally mapped $\rightarrow$ plane (or sphere).

the map in 2 steps

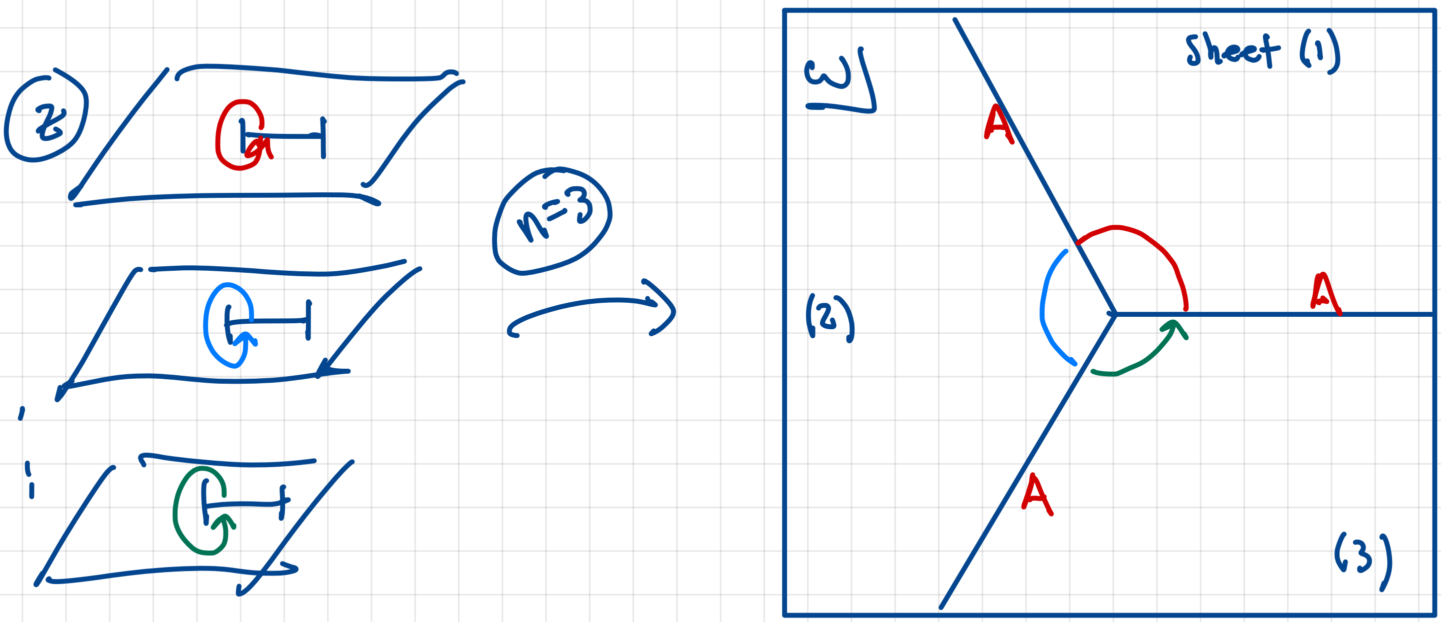
① $y = \dfrac{z - z_1}{z_2 - z}$



② $w = y^{1/n}$

So

$$w = \left(\frac{z - z_1}{z_2 - z} \right)^{1/n}$$



as $z - z_1 \to (z - z_1) e^{2\pi i}$, $w \to w e^{2\pi i/n}$

Success!

Now Calculate $\langle T(z) \rangle_{\text{replica mf}}$.

use general rel'n Weyl vs. conformal,

$$\langle \mathcal{O}(x) \cdots \rangle_{\substack{\Omega^2 dx^2 \\ = (dx')^2 \\ dz d\bar{z}}} = \langle \mathcal{O}'(x') \cdots \rangle_{dx^2} \quad , \quad \Omega = \left| \frac{\partial x'}{\partial x} \right|^{1/d}$$

$x' = z$

$\uparrow$
$dw d\bar{w}$

In our case: $x = w$, $x' = z$

$$\langle T(z) \rangle_{\substack{\text{replica mf} \\ dz d\bar{z}}} = \langle T'(z) \rangle_{dw d\bar{w}}$$

$$T'(z) = \left(\frac{\partial w}{\partial z} \right)^2 \left(T(w) - \frac{c}{12} \{ z, w \} \right)$$

$= \underline{\text{ex.}}$

$$= w'(z)^2 \, T(w) + \frac{\frac{c}{24} \left(1 - \frac{1}{n^2}\right) (z_1 - z_2)^2}{(z - z_1)^2 (z - z_2)^2}$$

And

$$\langle T(w) \rangle_{dw d\bar{w}} = 0 \quad \text{by transl./conf.}$$

$\Rightarrow$

$$\langle T(z) \rangle_{\substack{\text{replica} \\ \text{mf.}}} = \frac{1}{n} \frac{h_n \, (z_1 - z_2)^2}{(z-z_1)^2 (z-z_2)^2}$$

$$h_n := \frac{c}{24}\left(n - \frac{1}{n}\right)$$

$$\left(\begin{array}{l} \text{Recall on plane} \\ \langle T(z)\, \mathcal{O}(z_1,\bar{z}_1)\, \mathcal{O}(z_2,\bar{z}_2) \rangle_{pl} = \dfrac{h \, (z_1 - z_2)^2}{(z-z_1)^2 (z-z_2)^2} \end{array} \right)$$

Ward id.

extra from n sheets

$$\partial_{z_1} \log Z_{\text{replica}} = n \operatorname*{res}_{z \sim z_1} \langle T(z) \rangle_{\text{replica}} = -\frac{2h_n}{z_1 - z_2}$$

$$\hookrightarrow \frac{1}{\langle 1 \rangle_{\text{replica}}} \partial_{z_1} \langle 1 \rangle_{\text{replica}}$$

$$\bar{\partial}_{\bar{z}_1} \log Z_{\text{replica}} = -\frac{2h_n}{\bar{z}_1 - \bar{z}_2}$$

$$\Longrightarrow$$

$$\log \underbrace{Z_{\text{replica}, n}}_{\mathrm{Tr}\,\rho_A^n} = -4h_n \log |z_1 - z_2| + \text{const.}$$

$$\boxed{\mathrm{Tr}\,\rho_A^n = \alpha_n \times |z_1 - z_2|^{-4h_n}}$$

Normalize: $\mathrm{Tr}\,\hat{\rho}_A^n = \hat{\alpha}_n |z_1 - z_2|^{-4h_n}, \quad \hat{\alpha}_1 = 1$

Looks identical to $\langle \mathcal{O}\mathcal{O}\rangle$:

$$\mathrm{Tr}\,\hat{\rho}_A^{\,n} \equiv \left\langle \sigma_n(z_1)\,\sigma_{-n}(z_2) \right\rangle_{pl.}$$

$$\uparrow$$

this defines "twist operator"

= nonlocal primary of weights (h_n, h_n)

Entanglement Entropy

$$S_n = \frac{1}{1-n} \log \mathrm{tr}\,\hat{\rho}_A^{\,n}$$

$$= \frac{1}{1-n}\left(-4h_n \log|z_1-z_2| + \log \hat{d}_n\right)$$

$$S(\rho_A) = S_{n\to 1}$$

$$S(\rho_A) = \frac{c}{3}\log|z_1-z_2| + \text{const.}$$

$$\boxed{S(\rho_A) = \frac{c}{3}\log\frac{|z_1-z_2|}{\varepsilon}}$$

ε = UV cutoff.

(By dimensional analysis!)