14.

Quantum Entropy

Read: QGBH 18, 19

for more complete discussion see textbooks by Preskill or Wilde

Definition

$$S(\rho) = -\mathrm{tr}\, \rho \log \rho = -\sum_i \lambda_i \log \lambda_i$$

Basic properties

* $S \geq 0$

* $S(\rho) = 0$ iff ρ is pure,

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

$$S = -1 \log 1 - \sum 0 \log 0 = 0$$

* $S \leq S\left(\frac{1}{\dim \mathcal{H}} \mathbb{1}\right)$ $\quad$ ($= \infty$ in QFT!)

* if $V : \mathcal{H} \to \mathcal{H}'$ is an "isometry" $V^\dagger V = \mathbb{1}$, then

$$S(V \rho V^\dagger) = S(\rho)$$

Interpretation

$S(\rho_A) = \log$ (# auxiliary states required to purify ρ_A)

For $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$,

$$\rho_A = \mathrm{tr}_B\, \rho_{AB} \quad , \quad \text{etc.}$$

$S(\rho_A)$ = "entanglement entropy"

Caveat: only directly related to entanglement if ρ_{AB} is pure.

* subadditivity:

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

* Complementary subsystems:

If ρ_{AB} is pure, then $S(\rho_A) = S(\rho_B)$

Proof:

Schmidt: $|\psi\rangle = \sum_{i=1}^{N} \lambda_i \underbrace{|i\rangle_A |i\rangle_B}_{\text{orthonormal}}$, $\lambda_i \in [0,1]$

with $N \leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$

$\Rightarrow$ $$\rho_A = \sum_i \lambda_i^2 \, |i\rangle_A \, {}_A\langle i|$$

$$\rho_B = \sum_i \lambda_i^2 \, |i\rangle_B \, {}_B\langle i|$$

Same eigenvals $\Rightarrow$ $S_A = S_B$

* <u>Strong subadditivity</u> $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

$$S(B) + S(ABC) \leq S(AB) + S(BC)$$

Relative Entropy

$$S(\rho \| \sigma) = -\operatorname{tr} \rho \log \sigma + \operatorname{tr} \rho \log \rho$$

"measures distinguishability"

$$S(\rho \| \tfrac{1}{N} \mathbb{1}) = -\operatorname{tr} \rho \log \tfrac{1}{N} - S(\rho)$$
$$= \log \dim \mathcal{H} - S(\rho)$$

"measure of information content"

Monotonicity:

$$S(\rho_A \| \sigma_A) \leq S(\rho_{AB} \| \sigma_{AB})$$

Mutual Information

$$I(A,B)_\rho = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$$= S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

"measures correlation" (classical and quantum)

subadditivity: $I(A,B) \geq 0$

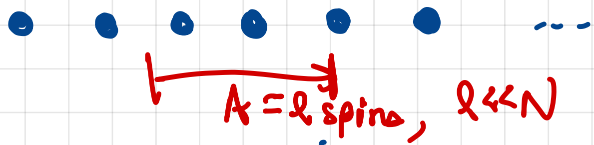
SSA $\quad I(A,B) \leq I(A,BC)$

This monotonicity of rel. ent:

$$S(\rho_{AB} \| \rho_A \otimes \rho_B) \leq S(\rho_{ABC} \| \rho_A \otimes \rho_{BC})$$

Subregion Entropy in QM

N spins in chain:



Periodic BC

$\mathcal{H}$: span$\left(|\sigma_1 \sigma_2 \cdots \sigma_N\rangle \,,\ \sigma_i \in \{0,1\} \right)$

classical/computational basis

$$|\Psi\rangle = \sum_{\{\sigma_i\}} \underbrace{\psi_{\sigma_1 \cdots \sigma_N}}_{2^N\ \mathbb{C}\text{ numbers}} |\sigma_1 \cdots \sigma_N\rangle$$

$A = \{$spins $k+1, \ldots, k+\ell\}$,

Suppose $\ell \ll N$

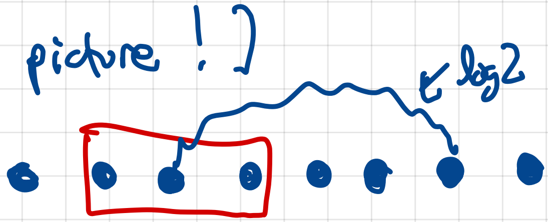
In most pure states,

$$\rho_A \approx \mathbb{1}_{2^\ell \times 2^\ell}$$

So

$$S(A) \sim \ell \log 2$$ "Volume Law" $S \sim$ Vol.

[draw picture!]



However, in the ground state of a local Hamiltonian, correlations are short range:

$$\langle \sigma_i \sigma_j \rangle \to 0 \quad \text{as} \quad |i-j| \to \infty$$

"a spin is only entangled w/ nearby spins"

Then

$$S(A) \to \begin{cases} \text{const.}\, D & \text{as} \quad |i-j| \to \infty \qquad \text{gapped} \\ \log \ell & \qquad \text{ungapped} \end{cases}$$

Higher dim'ns: In ground state,

$$S(A) \sim \text{Area}$$

or

$$\text{Area} \log R$$

Point Ground states of local Hamiltonians live in a very special corner of Hilbert space

"Matrix Product States"

Thm. Any 1d state w/ $S(A) \leq 2S_0$ as $|i-j| \to \infty$

can be written

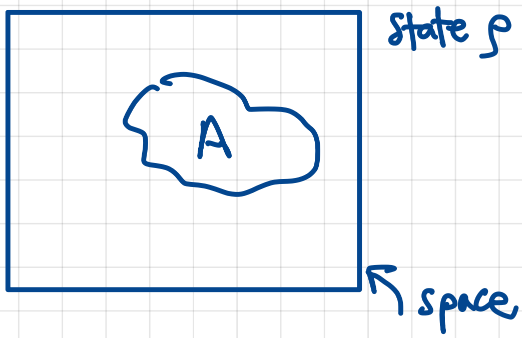
$$|\psi_0\rangle = \sum_{\{\sigma_i\}} M_{\sigma_1}^{i_1 i_2} M_{\sigma_2}^{i_2 i_3} M_{\sigma_3}^{i_3 i_4} \cdots M_{\sigma_N}^{i_{..} i_1} \, |\sigma_1 \cdots \sigma_N\rangle$$

where $i_n = 1 \ldots D = e^{S_0}$

ND^2 $\mathbb{C}$ numbers $\ll$ 2^N as $N \to \infty$

("Data compression" / RG.)

Subregion Entropy in QFT



$S(\rho_A)$ is UV divergent.

All states are vacuum in UV, so set $\rho = |0\rangle\langle 0|$

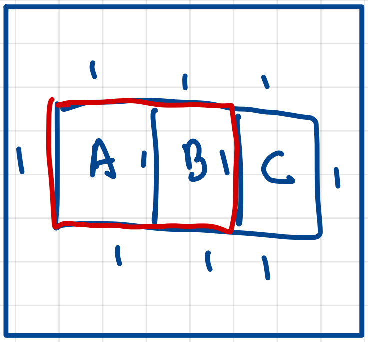
$$S(\rho_A) = \underbrace{\left(\frac{\#}{\varepsilon^{d-2}} + \frac{\#}{\varepsilon^{d-4}} \right)}_{\substack{\text{non-universal} \\ \text{(scheme dependent)}}} + \cdots + \underbrace{\# \log \varepsilon}_{\substack{\text{(even } d) \\ \text{any}}} + \text{finite}$$

$$\text{coefficients} = \int_{\partial A} \sqrt{h}\, F(K_{ab}, h_{ab})$$

$S(A) = S(B) \Rightarrow$ only even powers of $K \cup \nabla n$

We must be very careful to only talk about scheme-independent stuff in QFT!

Example:



Subadditivity:

$$S_A + S_B \geq S_{AB}$$

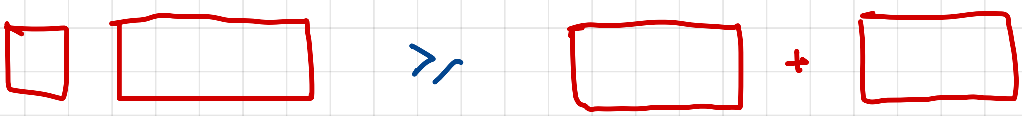
$$4\infty + 4\infty \geq 6\infty$$

$$8\infty \geq 6\infty$$

trivial!

But SSA:

$$S_B + S_{ABC} \geq S_{AB} + S_{BC}$$

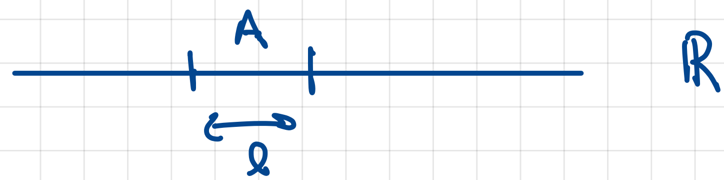


$$4 + 8 \geq 6 + 6$$

Nontrivial: constrains finite part.

Examples in 2d CFT (calc. later!)

Space:



In vacuum:

$$S(A) = \frac{c}{3} \log\left(\frac{\ell}{\varepsilon}\right) \qquad \left(+ \text{ non-univ. const}\right)$$

ε ← UV cutoff

"area" divergence

In thermal state:

$$S(A) = \frac{c}{3} \log\left[\frac{\beta}{\pi\varepsilon} \sinh\left(\frac{\pi \ell}{\beta}\right)\right]$$

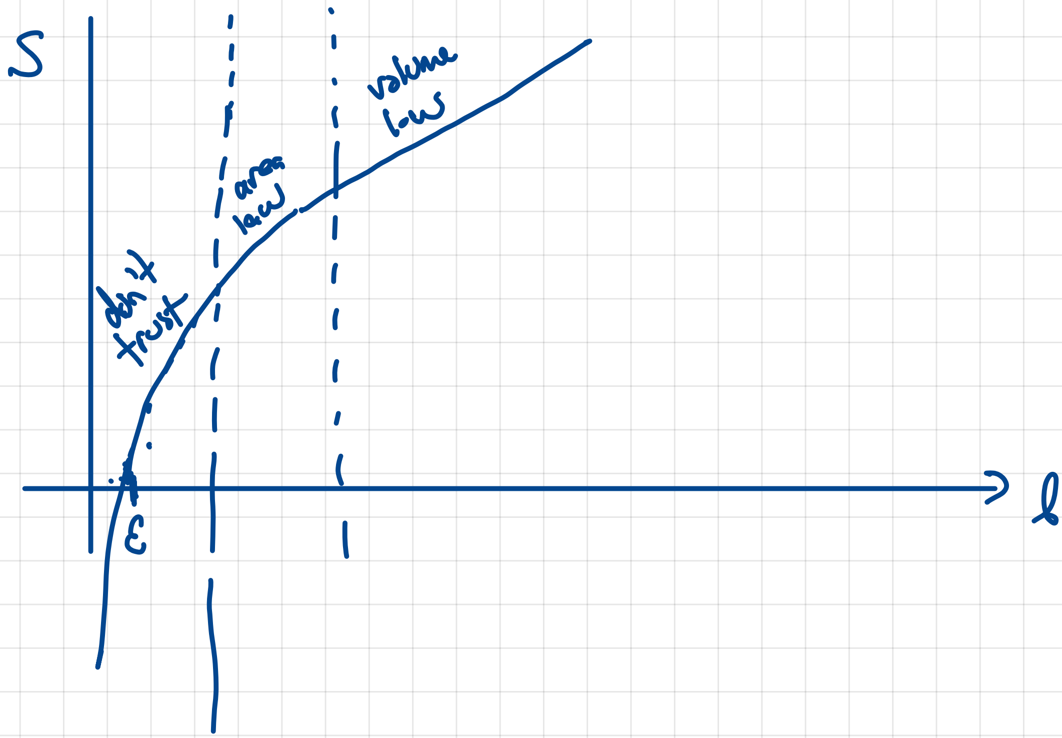
$$\ell \ll \beta \quad \rightarrow \quad \frac{c}{3} \log \frac{\ell}{\varepsilon}$$

$$\ell \gg \beta \quad \rightarrow \quad \frac{c}{3} \log\left[\frac{\beta}{\pi\varepsilon} \frac{1}{2} e^{\pi \ell/\beta}\right] \sim \frac{c}{3}\left(\frac{\pi \ell}{\beta}\right) + \text{const.}$$

"Volume law"

$S_{th}(\beta) \times \text{Vol}.$

in a thermal state,
extensive part of $S(\rho_A) \sim$ thermodynamic entropy



S
don't trust
area law
volume law
ε
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