13.

2d CFT

Read: Simmons-Duffin Phys 229 lecture notes
§ 15.1 - 15.8

Symmetries

In 2d, the conformal group is bigger.

$$ds^2 = dz\,d\bar{z} \quad , \qquad z = \sigma^1 + i\sigma^2$$

Conformal transformations :

$$z \to z' = f(z) \leftarrow \text{any holomorphic fn.}$$

$$\bar{z} \to \bar{z}' = \bar{f}(\bar{z})$$

Check:

$$ds^2 \to \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} dz\,d\bar{z}$$

$$= \Omega^2 \, dz\,d\bar{z} \quad \checkmark$$

Inf'ly,

$$z' = z + \varepsilon(z), \quad \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z})$$

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} a_n z^{n+1} \quad , \qquad \bar{\varepsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{a}_n \bar{z}^{n+1}$$

Thus [in vector language like higher dims]

$$x^\mu \to x^\mu + f^\mu$$

$$l_n = z^{n+1} \partial$$

$$\bar{l}_n = \bar{z}^{n+1} \bar{\partial}$$

Reality condition:

$$\sigma^{\mu\prime} \in \mathbb{R}$$

ie $\quad z' + \bar{z}', \quad i(z' - \bar{z}') \in \mathbb{R}$

Sets $\bar{a}_n = a_n^*$, but usually safe to think of these as independent !

Lie Brackets: $\quad [X, Y]_{\mathcal{L}} = \mathcal{L}_X Y$

$$[\xi_m, \xi_n]_{\mathcal{L}} = -(m-n)\, \xi_{m+n}$$

$$[\bar{\xi}_m, \bar{\xi}_n]_{\mathcal{L}} = -(m-n)\, \bar{\xi}_{m+n}$$

This is $2 \times \infty$ <u>real</u> generators.

In <u>general</u>, spacetime symmetries obey

$$\left[Q[\xi], Q[\xi'] \right] = Q_{-[\xi, \xi']} + \text{"central"}$$

Stress Tensor

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\partial T_{\bar{z}\bar{z}} + \bar{\partial} T_{z\bar{z}} = 0 \qquad (\partial = \partial_z)$$

$$\bar{\partial} T_{zz} + \partial T_{z\bar{z}} = 0$$

$$T^\mu_\mu = T_{z\bar{z}} = 0$$

Defn:

$$T = -2\pi T_{zz}$$

$$\bar{T} = -2\pi T_{\bar{z}\bar{z}}$$

$$\Theta = 2\pi T_{z\bar{z}}$$

$$\boxed{\bar{\partial} T = \partial \bar{T} = \Theta = 0}$$

$$T = T(z)$$

$$\bar{T} = \bar{T}(\bar{z})$$

Charges:

$$Q[\xi] = \int_{\text{space}} d\Sigma^\mu \, T_{\mu\nu} \, \xi^\nu$$

In radial quantization,

"space" = circle centered @ origin

$$L_n = Q[\varepsilon_n] = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z)$$

$$\bar{L}_n = \quad \ldots \quad = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z})$$

Invert:

$$T(z) = \sum_{n=-\infty}^{\infty} L_n\, z^{-n-2}$$

Polyakov anomaly action

Recall in curved b.g.,

$$\Theta = -\frac{c}{12} R$$

$$\langle \Theta \rangle_g = \frac{1}{2\pi} \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \log Z$$

use this to calculate $Z[g]$

First in conformal gauge:

$$g_{\mu\nu} = e^{2\rho} \delta_{\mu\nu}$$

$$\langle \Theta \rangle = -\frac{c}{6} \Box \rho$$

Weyl variation $\delta\rho$:

$$\delta \log Z[e^{2\rho} \delta_{\mu\nu}] = \frac{1}{2\pi} \int d^2z \sqrt{g} \langle \Theta \rangle_{e^{2\rho}\delta_{\mu\nu}} \delta\rho$$

$$= -\frac{c}{12\pi} \int d^2z \; \sigma \Box \rho$$

Integrate:

$$Z[e^{2\rho} \delta] = Z[\delta] \exp\left(\frac{c}{24\pi} \int d^2z \; (\partial_\mu \rho)^2 \right)$$

Any 2d metric can be put in conformal gauge. Therefore to find general $Z[g]$ need to find a covariant formula that reduces to this in conformal gauge. Polyakov did so:

Polyakov:

$$\boxed{Z[g] = Z[\delta]\, e^{-S_P[g]}}$$

$$S_P[g] = \frac{c}{96\pi} \int d^2x \sqrt{g}\, R \frac{1}{\Box} R$$

i.e.

$$= \frac{c}{96\pi} \int d^2x \int d^2y \sqrt{g(x)} \sqrt{g(y)}\, R(x)\, G(x,y)\, R(y)$$

$$\text{w/} \quad \Box G(x,y) = \frac{1}{\sqrt{g(x)}} \delta^{(2)}(x-y)$$

Equivalently,

$$S_P[g] = \frac{c}{48\pi} \int d^2x \sqrt{g(x)} \left(\frac{1}{2} (\partial\phi)^2 + \phi R \right)$$ "Liouville"

eval'd on-shell: $\Box\phi = R$

Conclude:

S_p encodes all dependence on $g_{\mu\nu}$, and thus all

$$\langle T_{\mu\nu}(x_1)\, T_{\alpha\beta}(x_2) \cdots \rangle \quad \text{determined}$$

solely by c!

This is not true in $d>2$, where there is honest-to-goodness dynamics in $g_{\mu\nu}$.

Ex:

$$\langle T(z)\, T(0) \rangle = \frac{c}{2z^4}$$

Conformal Transf. of $T_{\mu\nu}$

Recall that the conformal transformation law can be obtained from a special case of Weyl that preserves the mf.

Weyl:

$$\langle T(z) \rangle_{e^{2\rho}\delta_{\mu\nu}} = \frac{4\pi}{\sqrt{g(z)}} \frac{\delta}{\delta g_{\bar{z}\bar{z}}(z)} \overbrace{\left(-S_p[g]\right)}^{\log Z}$$

$$\delta S_p(g) \sim \delta \int (\partial\phi)^2 + \phi R \Big|_{\phi = \Box R}$$

$$\sim -\frac{1}{2} \int \sqrt{g}\, T_\phi^{\mu\nu} \delta g_{\mu\nu} \Big|_{\phi = \Box R}$$

where

$$T^\phi_{\mu\nu} = \frac{c}{24\pi}\left[\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\left(\Box\phi - \frac{1}{4}(\nabla\phi)^2\right)\right]$$

Now set $\Box\phi = R[e^{2\rho}\delta_{\mu\nu}] = -2\Box\rho$, $\phi = -2\rho$

$\Rightarrow$

$$\langle T \rangle = -2\pi T_{zz}$$

$$= \frac{c}{12}\left[2(\partial\rho)^2 - 2\,\partial^2\rho \right] \qquad (g_{zz}=0)$$

$$\boxed{\langle T \rangle_{e^{2\rho}\delta_{\mu\nu}} = \frac{c}{6}\left[(\partial\rho)^2 - \partial^2\rho \right]}$$

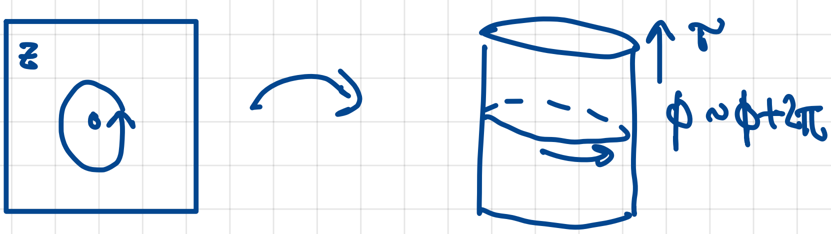
Example : Cylinder

$$ds^2_{cyl} = \frac{1}{|z|^2}\, dz\, d\bar{z} \quad , \qquad z \in \mathbb{C}$$

check:

$$z = e^{i\omega} \quad , \quad \omega = \phi + i\tau$$

$$\bar{z} = e^{-i\bar{\omega}} \quad , \quad \bar{\omega} = \phi - i\tau$$



$$ds^2_{cyl} = d\tau^2 + d\phi^2 , \qquad \tau \in \mathbb{R} , \quad \phi \sim \phi + 2\pi$$

$$= d\omega\, d\bar{\omega}$$

$$\omega \sim \omega + 2\pi$$

$$\rho = -\frac{1}{2}\log(z\bar{z})$$

$$\langle T(z) \rangle_{\frac{1}{|z|^2} dz d\bar{z}} = -\frac{c}{24 z^2}$$

$$T(w) \equiv -2\pi T_{ww}(w)$$

Now Coord change:

$$\langle T(w) \rangle_{\substack{dw d\bar{w} \\ w \sim w+2\pi}} = \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} \langle T(z) \rangle_{\frac{1}{|z|^2} dz d\bar{z}}$$

$$\frac{\partial z}{\partial w} = e^{iw} = iz$$

$$\boxed{\langle T(w) \rangle_{\substack{dw d\bar{w} \\ w \sim w+2\pi}} = \frac{c}{24}}$$

Caution: Coord. Changes in CFT are same as always, no funny factors, anomalies, etc. !

Charges:

$$Q[\partial_w] = L_0 - \frac{c}{24}$$

$$Q[\partial_{\bar{w}}] = \bar{L}_0 - \frac{c}{24}$$

Hamiltonian

$$H_{cyl} = Q[\partial_\tau] = D - \frac{c}{12}$$

$\Rightarrow$ Vacuum has Casimir energy

$$E_{vac} = -\frac{c}{12}$$

Conformal transf. law

Recall that special Weyl transf $\to$ conformal transf.

Scalars:

$$\langle \mathcal{O}(x) \cdots \rangle_{(dx')^2} = \langle \mathcal{O}'(x') \cdots \rangle_{dx^2}$$

$\uparrow$

$\Omega^2 dx^2$ with $\Omega = \left| \frac{\partial x'}{\partial x} \right|^{1/d}$

We'll use this to find $T'(x')$

2d:

$x = z$, $x' = w$, $z(w)$, $\bar{z}(\bar{w})$

$$\Omega^2 = e^{2\rho} = \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}}$$

$$\rho = \frac{1}{2} \log w'(z) \bar{w}'(\bar{z})$$

Special Weyl (from above):

$$\langle T(z) \rangle_{\Omega^2 dz d\bar{z}} = \frac{c}{6} \left((\partial\rho)^2 - \partial^2 \rho \right)$$

(plug in ρ above)

$$= -\frac{c}{12} \{w, z\}$$

where we defined

"Schwarzian Derivative"

$$\{ f(z), z \} = \frac{f'''}{f'} - \frac{3}{2} \frac{(f'')^2}{(f')^2}$$

The above rel'n Weyl → conformal, now for spinning operator, is

$$\left(\frac{\partial w}{\partial z}\right)^2 \langle T(z) \cdots \rangle_{\Omega^2 dz d\bar{z}} = \langle T'(w) \cdots \rangle_{dz d\bar{z}}$$

⇒ Conformal Transf. Law:

$$\boxed{T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12} \{ w, z \} \right]}$$

Satisfies, for example

$$\langle T(w) \rangle_{cyl} = \langle \text{r.h.s.} \rangle_{pl.}$$

Equivalently

$$\delta_\varepsilon T(z) = -2T\partial\varepsilon - \varepsilon\partial T - \frac{c}{12}\partial^3\varepsilon$$

Equivalently

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

Equivalently

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m+n,0}}$$

Virasoro algebra

Primaries

Defn.

$$[L_0, \mathcal{O}(0)] = h\,\mathcal{O}(0)$$

$$[\bar{L}_0, \mathcal{O}(0)] = \bar{h}\,\mathcal{O}(0)$$

$$\ell_0 \pm \bar{\ell}_0 = z\partial \pm \bar{z}\bar{\partial}, \quad \text{so}$$

$$\Delta = h + \bar{h}, \quad \text{helicity } \ell = h - \bar{h}$$

Conformal transformation

$$\mathcal{O}(z,\bar{z}) \to \mathcal{O}'(z,\bar{z}) = \left(\frac{\partial z}{\partial z'}\right)^{-h} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{-\bar{h}} \mathcal{O}(z',\bar{z}')$$

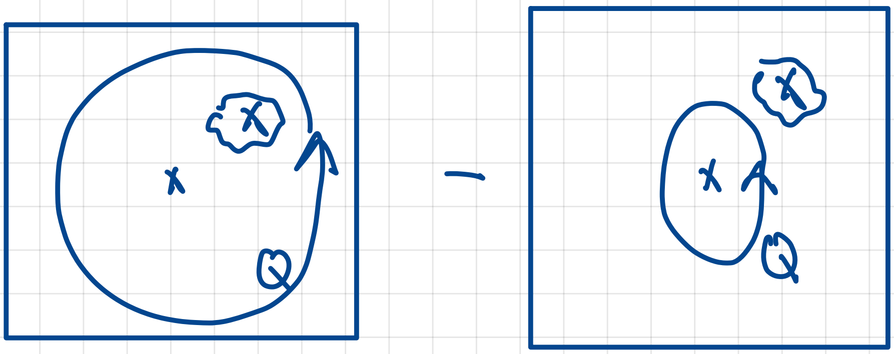
$T_{\mu\nu}$ is not primary in 2d

it's quasiprimary - $SO(2,2)$

Ward Identity (Di Francesco 5.46)

$$[Q[\varepsilon], X] = \delta_\varepsilon X$$

$\parallel$



$$\delta_\varepsilon \langle X \rangle = \int d^2x \, \partial_\mu \langle T^{\mu\nu}(x) \varepsilon_\nu(x) X \rangle$$

anything

conformal transf.

(Gauss)

$$\delta_\varepsilon \langle X \rangle = -\frac{1}{2\pi i} \oint dz \, \varepsilon(z) \langle T(z) X \rangle$$

$$+ \frac{1}{2\pi i} \oint d\bar{z} \, \bar{\varepsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle$$

$$\boxed{\delta_\varepsilon X(w) = \operatorname*{res}_{z\sim w} \varepsilon(z) T(z) X(w,\bar{w}) + \operatorname*{res}_{\bar{z}\sim\bar{w}} \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}) X(w,\bar{w})}$$

See Polchinski eqn. 2.6.15

Example

translation $z \to z+a \Rightarrow$ conf. transf. $\varepsilon(z) = a$

therefore

$$\operatorname*{res}_{z\sim w} T(z)\mathcal{O}(w) = \delta_{\varepsilon=a}\mathcal{O}(w)$$

$$= \partial\mathcal{O}(w)$$

dilatation $\Rightarrow$ $\operatorname*{res}\, zT(z)\mathcal{O}(w) = h\mathcal{O}(w)$

$$\therefore \quad \boxed{T(z)\mathcal{O}(w) \sim \underbrace{\cdots}_{\text{vanish for primary}} + \frac{h}{(z-w)^2}\mathcal{O}(w) + \frac{1}{z-w}\partial\mathcal{O}(w) + \cdots}$$

$\therefore$ for primaries,

$$\langle T(z)\,\mathcal{O}(w_1)\,\mathcal{O}(w_2)\rangle$$

$$= \frac{h}{(z-w_1)^2}\langle \mathcal{O}(w_1)\,\mathcal{O}(w_2)\rangle$$

$$+ \frac{1}{z-w_1}\langle \partial\mathcal{O}(w_1)\,\mathcal{O}(w_2)\rangle$$

$$+ (w_1 \leftrightarrow w_2)$$

$$+ \cdots$$

w/ <u>NO</u> other singularities $z \in \mathbb{C}$

<u>regularity @ $z=0$</u> requires $T(z) \sim \text{const} + z + \cdots$

<u>" @ $z=\infty$:</u>

$$x = \frac{1}{z} \qquad T_{xx}(x) = \left(\frac{dz}{dx}\right)^2 T_{zz}(z) = x^{-4}\,T_{zz}(z)$$

reg. @ $x=0$ $\Rightarrow$ $T_{xx}(x) \sim$ const.

$\Rightarrow$ $T_{zz}(z) \sim x^4 \sim \frac{1}{z^4}$ as $z \to \infty$.

$\Rightarrow$

$$\boxed{\langle T(z)\,\mathcal{O}(w_1)\,\mathcal{O}(w_2)\rangle = \frac{h\,(w_1-w_2)^2}{(z-w_1)^2\,(z-w_2)^2}}$$