10.

Gravitational Path Integrals

Read: QGBH §6

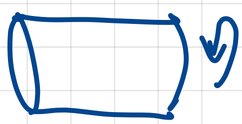
In QFT,

$$Z = \int D\phi \, e^{-I_E[\phi]}$$

all fields except $g_{\mu\nu}$

for example

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

$=$  $\beta = \displaystyle\int_{\text{cyl.}} D\phi \, e^{-I_E[\phi]}$

So in QFT, we fix M, and integrate over ϕ.

In gravity, we need to also sum over geometries.

In quantum gravity,

$$Z = \text{“} \int D\phi \, Dg_{\mu\nu} \; e^{-I_E[g,\phi]} \text{”}$$

We cannot do the integral of course - not renormalizable.

We will nonetheless spend the rest of the course (lives?) trying to make sense of this integral.

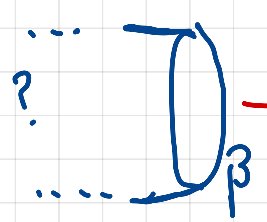
We do not specify Mf., but we do set boundary conditions.

Ex.

$$Z(\beta) = \boxed{\begin{array}{l} \int D\phi \, Dg \; e^{-I_E} \\ \text{subject to} \\ ds^2\Big|_{\substack{\text{near} \\ \text{infinity}}} \approx d\tau^2 + d\vec{x}^2 + \text{subleading} \\ \text{with} \quad \tau \sim \tau + \beta \end{array}}$$

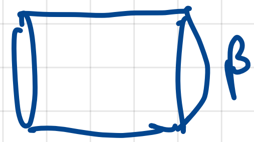
this cartoon represents this path integral

i.e. "asymptotically cylinder of proper size β"

i.e. $Z(\beta) =$ 

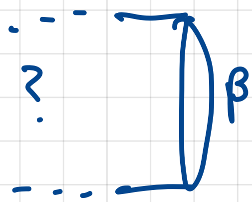
* Important difference *

In QFT, we derived

$$\mathrm{Tr}\, e^{-\beta H} = $$ 

(by slicing the path integral, inserting a complete set of states.)

In gravity, we postulate

$$\mathrm{Tr}\, e^{-\beta H} = $$ 

We cannot derive this by inserting a complete set of states. We guess this formula, and check that it gives reasonable and consistent results.

Semiclassical Approximation

As I said we do not know how to do the path integral, so how do we proceed?

$$\int Dg D\phi \, e^{-I_E[g,\phi]}$$

$$\approx \sum_{\text{saddles}} e^{-I_E[\underbrace{\bar{g},\bar{\phi}}_{\text{Classical solution (saddlepoint)}}]} + I^{(1)} + I^{(2)} + \cdots$$

$$G_N^{-1} \qquad\qquad G_N^{0} \qquad\qquad G_N \qquad \cdots$$

Suppose $\bar{\phi} = 0$. (so we can think about gravity alone)

then

$$\boxed{Z(\beta) \approx e^{-I_E(\bar{g})}}$$

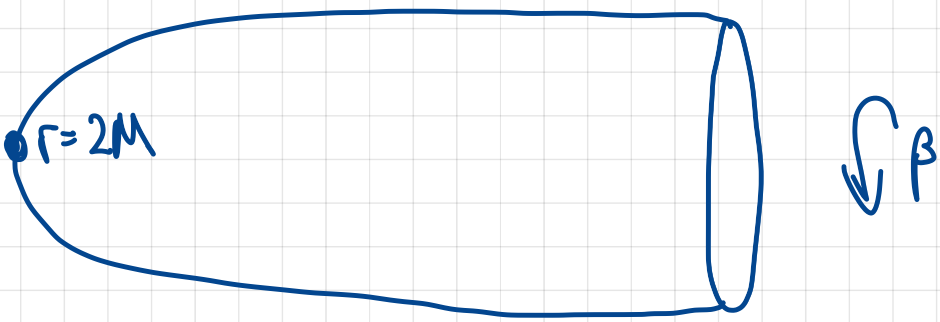
where $\bar{g}_{\mu\nu}$ is a solution of EE's with

$$d\bar{s}^2 \approx d\tau^2 + d\vec{x}^2 + \cdots \quad @ \ \infty$$
$$\tau \sim \tau + \beta$$

Do you know any solutions like that?

Yes! Euclidean Schwarzschild BH!

$\bar{g}_{\mu\nu}$ = Euclidean BH!



(This obeys our boundary condition and solves EE.)

Recall free energy

$$\beta F = -\log Z(\beta)$$

$$= I_E[\bar{g}]$$

On-shell action

As we will see, all the usual properties of a thermodynamic free energy apply here.

4d Schwarzschild

We will now calculate the leading term for an example.

Einstein action

$$I_E = \frac{-1}{16\pi} \int_M d^d x \sqrt{g}\, R - \underbrace{\frac{1}{8\pi} \int_{\partial M} d^{d-1} x \sqrt{h}\, K}_{\text{"GHY"}}$$

$$\left(+ \text{ counterterms/subtractions} \right)$$

Why GHY?

$$\delta \int_M \sqrt{g} R \sim \int_M \sqrt{g}\, \underbrace{G^{\mu\nu}}_{\text{eom}} \delta g_{\mu\nu}$$

$$+ \int_{\partial M} \sqrt{h} \left[(\quad) \delta g_{\mu\nu} + (\quad) \partial_n \delta g_{\mu\nu} \right]$$

BAD!

Dirichlet boundary condition

$$g|_{\partial M} = \gamma \,, \quad \delta g|_{\partial M} = 0$$

$$\Rightarrow \quad \delta\left(\int \sqrt{g}\, R\right) \neq 0 \quad \text{on solutions to EE!}$$

GHY cancels this term, so

$$\delta I_E = \int_M (\text{eom})\, \delta g + \int_{\partial M} (\ )\, \delta g$$

$$= 0 \quad \text{on-shell.}$$

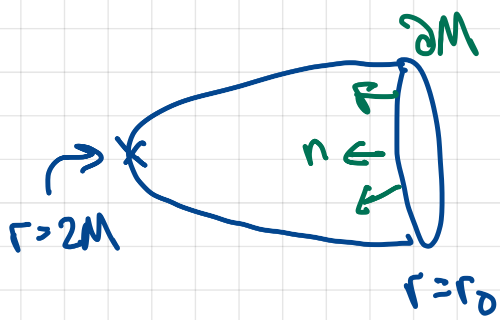
General words of wisdom: boundary terms in the action should be appropriate to your choice of boundary condition. The boundary condition is part of the definition of the theory.

Evaluate I_E for Euclidean BH

$$ds^2 = \left(1-\frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2$$

$$\tau \sim \tau + 8\pi M$$

restrict $r < r_0$



$\Rightarrow$ $R = 0$

Calculate K

Unit Normal on ∂M:

$n \propto \partial_r$ with $n^2 = 1$

$$n = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_r$$

Induced metric on ∂M

$h_{ij} = g_{\mu\nu}$ with $i,j = \tau, \theta, \phi$

and $r = r_0$

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = \nabla_{(i} n_{j)}$$

$$= \frac{1}{2} n^\mu \partial_\mu h_{ij}$$

$$K = h^{ij} K_{ij}$$

$\Longrightarrow$

$$\int_{\substack{\partial M \\ r=r_0}} d\tau\, d\theta\, d\phi \sqrt{h}\, K = \int d\tau \left(8\pi r_0 - 12\pi M + \mathcal{O}\left(\frac{1}{r_0}\right)\right)$$

$$= \infty \quad \text{as} \quad r_0 \to \infty \,!$$

"counterterm"

shift $I_E \to$

$$\int \sqrt{h}\,(K - K_0)$$

$\uparrow$
curvature of
same ∂M embedded
in Minkowski

ie, let

$$ds^2_{aux} = \left(1 - \frac{2M}{r_0}\right) d\tau^2 + dr^2 + r^2 d\Omega^2$$

then K_0 = extrinsic curv. of $r = r_0$ in this mf.

$$\int \sqrt{h}\,(K-K_0) = \int d\tau \left(-4\pi M + \mathcal{O}\left(\frac{1}{r_0}\right) \right)$$

$$= -4\pi M \beta \qquad (r_0 \to \infty)$$

$$\beta = 8\pi M$$

$\Rightarrow$

$$I_E = -\frac{1}{8\pi} \, {-4\pi} \, \frac{\beta}{8\pi} \, \beta$$

$$= \frac{1}{16\pi} \beta^2 =$$

$$\boxed{Z(\beta) \approx \exp\left(-\frac{1}{16\pi} \beta^2 \right)}$$

Check thermo:

recall $S = \frac{1}{4}\text{Area} = 4\pi M^2$

$$Z = e^{-\beta F}$$

$$F = E - TS$$

$$= M - \frac{1}{8\pi M} 4\pi M^2 = \frac{1}{2} M$$

so $\beta F = \frac{1}{2} M \beta = \frac{1}{16\pi}\beta^2$ ✓

$$\langle E \rangle_\beta = -\partial_\beta \log Z$$

$$= \frac{1}{8\pi}\beta$$

$$= M$$ ✓

$$S = (1 - \beta\partial_\beta)\log Z$$

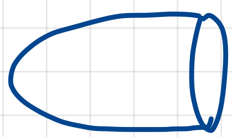
= ✓

Quantum Corrections

To all orders in perturbation theory:

$$\boxed{Z(\beta) \approx \sum_{\text{saddles}} e^{-I_E(\bar{g}_{\mu\nu})} \underbrace{Z_{QFT}(\bar{g}_{\mu\nu})}_{\substack{\text{ordinary QFT path integral} \\ \text{in the metric } \bar{g}_{\mu\nu} \\ \text{(treating } \delta g_{\mu\nu} \text{ as quantum field)}}}}$$

Phase transitions!

$$Z_{QFT}(\bar{g}_{\mu\nu}) = \text{QFT P.I. on}$$ 

$$= \text{Tr } e^{-\beta H_{QFT}}$$

This is the path integral we were talking about in our discussion of Hawking radiation.

At 1-loop:

1) Expand $I_E[g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},\ \phi = \bar{\phi} + \delta\phi]$ to quadratic order

2) Gauge-fix

3) Integrate $\int Dh_{\mu\nu} D\delta\phi \rightarrow \prod(\text{determinants})$

(Note: Instabilities of hot flat space)