the coefficient is fixed by the conformal Ward identity to be

$$\lambda_T \sim \frac{\Delta_O \Delta_\psi}{c} \,, \tag{2.12}$$

which is obviously positive. But in slightly different situations discussed below the constraints obtained by identical reasoning are nontrivial.

6. The constraint can also be stated using the maximum modulus principle: the magnitude of an analytic function in a region D is bounded by the maximum magnitude on ∂D. There is simply no way for a function of the form (2.6) to be analytic inside the semicircle in (2.8), bounded by 1 on the real line, and have $\lambda < 0$. This can be easily checked by finding the maximum of this function along the semicircle with $|\sigma| = R$, and comparing to the maximum along the semicircle $|\sigma| = R - \delta R$, which must be smaller. Similar reasoning rules out the possibility that the dominant exchange has $\ell > 2$, because in this case both choices of sign violate the maximum modulus principle. This version of the argument is inspired by the 'signalling' argument in [6] and the chaos bound [41].

Note that we do not assume causality; we use the conformal block expansion and reflection positivity to derive causality, then apply this result to derive the log bound. If we had simply assumed causality then the argument for log bounds would be significantly shorter.

We go through all of these steps in great detail in the rest of the paper.

3 Causality review

Causality requires commutators to vanish outside the light-cone:[2]

$$[O_1(x), O_2(0)] = 0, \qquad x^2 > 0, \qquad x \in R^{d-1,1} \,, \tag{3.2}$$

[2] Here is the standard argument for (3.2): The theory cannot be quantized in a way consistent with boost invariance if (3.2) is violated. To see this, add a local perturbation to the Hamiltonian, $H_{int} = \lambda O_2(x,t)\delta(x)\delta(t)$, and calculate in the interaction picture

$$\begin{aligned} \langle \Omega | O_1(x) | \Omega \rangle &= \langle 0 | e^{i \int_{-\infty}^t H_{int} dt} O_1(x) e^{-i \int_{-\infty}^t H_{int} dt} | 0 \rangle \\ &= \langle 0 | O_1(x) | 0 \rangle + \lambda \Theta(t) \langle 0 | [O_1(x), O_2(0)] | 0 \rangle + \cdots \,. \end{aligned} \tag{3.1}$$

For spacelike separation, the step function $\Theta(t)$ is not invariant under boosts, so different coordinate systems disagree about the $O(\lambda)$ term if it is non-zero. The same argument can be repeated in any state, so (3.2) holds as an operator equation.

where $O_{1,2}$ are local operators inserted in Minkowski space. In this section we will review how this requirement is encoded in the analytic structure of correlation functions, first in a general Lorentz-invariant QFT and then in CFT. This is an informal derivation of the position-space $i\epsilon$ prescription stated in standard references, for example the textbook by Haag [9].

3.1 Euclidean and Lorentzian correlators

In a general Lorentz-invariant QFT, consider the Euclidean correlator on a plane,

$$G(x_1, \ldots, x_n) = \langle O_1(x_1) \ldots O_n(x_n) \rangle \ . \tag{3.3}$$

This is a single valued, permutation invariant function of the positions

$$x_i = (\tau_i, x_i^1, \ldots, x_i^{d-1}) \in R^d \ . \tag{3.4}$$

Here τ is a direction, chosen arbitrarily, that will play the role of imaginary time. G is analytic away from coincident points, and has no branch cuts as long as all n points remain Euclidean. This reflects the fact that in Euclidean signature, operators commute:

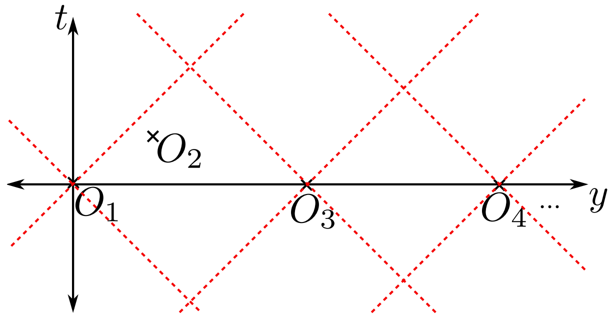
$$[O_1(x_1),\ O_2(x_2)] = 0, \qquad x_{1,2} \in R^d \ , \qquad x_1 \neq x_2 \ . \tag{3.5}$$

Lorentzian correlators can be computed (or defined) by analytically continuing $\tau_i \to it_i$, with t_i real. As functions of the complex τ_i, the correlator has an intricate structure of singularities and branch cuts, leading to ambiguities in the analytic continuation. Each choice that we make in the analytic continuation translates into a choice of operator ordering in Lorentzian signature, so these ambiguities are responsible for non-vanishing commutators. All of the Lorentzian correlators are analytic continuations of each other.[3]

For instance, suppose we aim to compute the Lorentzian correlation function with all t_i's zero except t_2, and displacement only in the direction $x^1 \equiv y$, pictured in

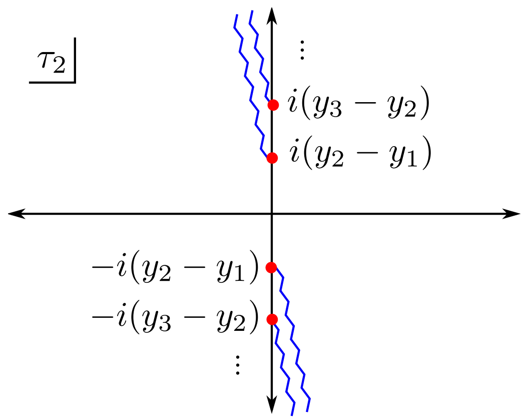
[3] This is simple to prove: the Lorentzian correlators with various orderings are equal when points are spacelike separated, so it is a standard fact of complex analysis (the edge-of-the-wedge theorem) that they must all be related by analytic continuation in the positions.

Lorentzian signature as follows:



(3.6)

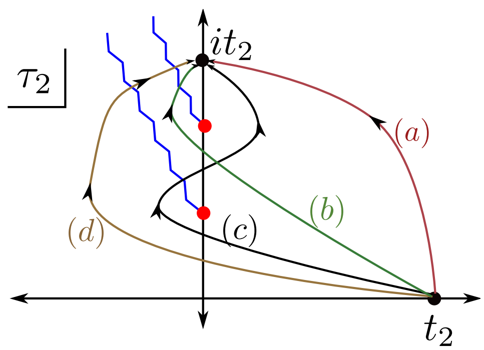
The correlator, viewed as a function of complex τ_2 with all other arguments held fixed, has singularities along the imaginary-τ_2 axis where O_2 hits the light cones of the other operators:



(3.7)

In an interacting theory, these singularities (red dots) are branch points, and we will orient the branch cuts (blue) so that they are 'almost vertical' as in the figure. In order to compute the correlator when O_2 is timelike separated from other operators, we need to continue from the point $\tau_2 = t_2$ on the positive real axis to the point $\tau_2 = it_2$ on the imaginary axis, which is above some light-cone singularities. Each time we pass a singularity, we must choose whether to pass to the right or to the left. Assume without loss of generality that $t_2 > 0$. Then passing to the right of a singularity puts the operators into time ordering in the resulting Lorentzian correlator, and passing to the left puts the operators in anti-time-order.

For example, suppose O_2 is in the future light cones of O_1 and O_3, but is spacelike separated from other operators. Then, starting from the single-valued Euclidean correlator, we can choose to go to Lorentzian signature along four different contours:



(3.8)

By choosing a contour, we mean that the analytic continuation is done in a way that is continuous along the given contour. These correspond, respectively, to the Lorentzian correlators

$$
\begin{array}{ll}
(a) & \langle O_2 O_1 O_3 \cdots \rangle = \langle T[O_1 O_2 O_3 \cdots] \rangle \\
(b) & \langle O_3 O_2 O_1 \cdots \rangle \\
(c) & \langle O_1 O_2 O_3 \cdots \rangle \\
(d) & \langle O_1 O_3 O_2 \cdots \rangle \ .
\end{array}
$$

(a) is fully time-ordered, (d) is fully anti-time-ordered, and the other two are mixed. If more than two operators were timelike-separated, then we would also need to worry about the ordering of the various branch cuts with respect to each other.

This recipe is motivated by the following observation. The branch cuts appear when operators become timelike separated, so to get a reasonable Lorentzian theory, the commutator must be equal to the discontinuity across the cut. This implies that, for example, the function defined along contour (a) in (3.8) differs from the function defined along (b) by adding a commutator, $[O_2, O_3]$. Combined with the fact that all Lorentzian correlators must continue to the (same) Euclidean correlator when operators are spacelike separated, this essentially fixes the prescription to what we have just described. See [9] and below for references to a full derivation.

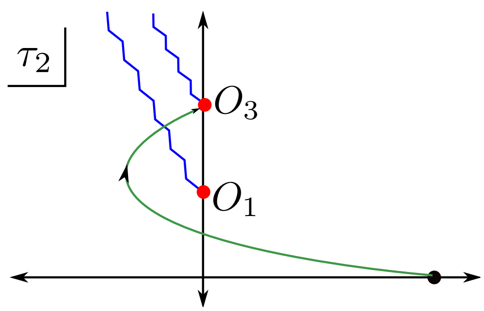
3.2 Causality

In order to diagnose whether a theory is causal, the actual value of the commutator is not needed – the only question is where it is non-zero. The answer, in the language of the analytically continued correlation functions, is that the commutator becomes non-zero when we encounter a singularity in the complex time plane and are forced to chose a contour.

The Euclidean correlator is singular only at coincident points. This immediately leads to a causal Lorentzian correlator on the first sheet of the τ_2 plane, which is the sheet pictured in (3.7). To see this, note that a singularity at $x^2 = 0$ in Euclidean continues to a singularity at $x^2 = 0$ in Lorentzian, which is obviously on the light cone. Thus, in the configuration discussed above, $\langle [O_2, O_1] O_3 \cdots \rangle$ and $\langle [O_2, O_3] O_1 \cdots \rangle$ are manifestly causal: they become non-zero at the branch points drawn in that figure, which start precisely at the light cones. However, as we pass onto another sheet by crossing a branch cut, singularities could move. For example, the commutator

$$\langle O_1 [O_2, O_3] \cdots \rangle \tag{3.9}$$

becomes non-zero when we encounter the O_3 singularity along this contour:



. (3.10)

It is not at all obvious that this singularity is at the O_3 lightcone, $\tau_2 = i(y_3 - y_2)$. If, as we pass through the O_1 branch cut, this $O_2 \to O_3$ singularity shifts upwards along the imaginary axis, then the theory exhibits a time delay. If it shifts downwards, then the commutator becomes non-zero earlier than expected (as a function of t_2) and the theory is acausal.

To summarize: Starting from a Euclidean correlator, causality on the first sheet is obvious. The non-trivial statement about causality is a constraint on how singularities in the complex-τ plane move around as we pass through other light-cone branch cuts.

3.3 Reconstruction theorems and the $i\epsilon$ prescription

The Osterwalder-Schrader reconstruction theorem [8] states that well behaved Euclidean correlators, upon analytic continuation, result in Lorentzian correlators that obey the Wightman axioms. The definition of a well behaved Euclidean correlator is (i) analytic away from coincident points, (ii) $SO(d)$ invariant, (iii) permutation invariant, (iv) reflection positive, and (v) obeying certain growth conditions. Reflection positivity is the statement that certain correlators are positive and is discussed more below.

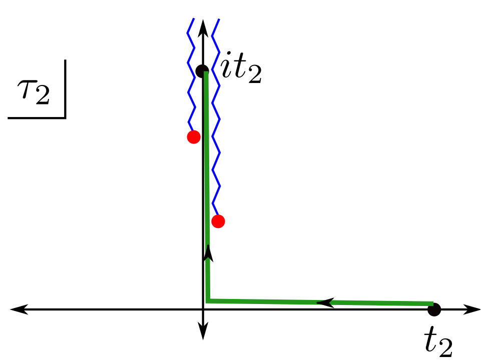
Lüscher and Mack extended this result to conformal field theory defined on the Euclidean plane, showing that the resulting theory is well defined and conformally invariant not only in Minkowski space but on the Lorentzian cylinder [44].

A byproduct of these reconstruction theorems is a simple $i\epsilon$ prescription to compute Lorentzian correlators, with any ordering, from the analytically continued Euclidean correlators:

$$\langle O_1(t_1,\vec{x}_1)O_2(t_2,\vec{x}_2)\cdots O_n(t_n,\vec{x}_n)\rangle = \lim_{\epsilon_j\to 0}\langle O_1(t_1-i\epsilon_1,\vec{x}_1)\cdots O_n(t_n-i\epsilon_n,\vec{x}_n)\rangle \quad (3.11)$$

where the limit is taken with $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$. The correlator on the rhs is analytic for any finite ϵ_k obeying these inequalities, which also confirms that the singularities we have been discussing always lie on the imaginary axis.

This $i\epsilon$ prescription is identical to our discussion above. It shifts the branch cuts to the left or right of the imaginary axis, and this enforces the contour choices that we described. For example contour (c) in (3.8) corresponds to the $i\epsilon$ prescription



, (3.12)

where the $i\epsilon$'s move the lightcone singularities off the imaginary-τ_2 axis as indicated.[4]

[4]Note that inserting $i\epsilon$'s into the correlator is meaningless unless we also specify the positions of all

In principle, the reconstruction theorem completely answers the question of when Euclidean correlators define a causal theory. Our point of view, however, will be that we assume only some limited information about the CFT — for example, there is some light operator of a particular dimension and spin, exchanged in a four-point function, perhaps with some particular OPE coefficients — and we want to know whether this is compatible with causality. This limited data may or may not come from a full QFT obeying the Euclidean axioms. The reconstruction theorem does not answer this type of question in any obvious way. In other words, the reconstruction theorem tells us that causality violation in Lorentzian signature must imply some problem in Euclidean signature, but we want to track down exactly what that problem is.

3.4 Examples

Conformal 2-point function

The Euclidean 2-point function in CFT is $(\tau^2+x^2)^{-\Delta}$. This is single-valued in Euclidean space, since the term in parenthesis is non-negative. Using the $i\epsilon$ prescription, the Lorentzian correlators for $t_1 > x_1$ are

$$\langle O(t_1,x_1)O(0,0)\rangle = \exp\left(-\Delta\log(-(t_1-i\epsilon)^2+x_1^2)\right) = e^{-i\pi\Delta}(t_1^2-x_1^2)^{-\Delta}\ , \qquad (3.13)$$

and

$$\langle O(0,0)O(t_1,x_1)\rangle = \exp\left(-\Delta\log(-(t_1+i\epsilon)^2+x_1^2)\right) = e^{i\pi\Delta}(t_1^2-x_1^2)^{-\Delta}\ , \qquad (3.14)$$

where we placed the branch cut of log on the negative real axis, as in (3.12).

Alternatively, in the language of paths instead of $i\epsilon$'s, we start from the Euclidean correlator $(z\bar{z})^{-\Delta}$, where $z = x + i\tau$, $\bar{z} = x - i\tau$. To find the time-ordered Lorentzian correlator we set $\tau = t_2 e^{i\phi}$ and follow the path $\phi \in [0, \pi/2]$. The result agrees with (3.13). The anti-time-ordering path goes the other way around the singularity at $z = 0$, so it differs by $z \to ze^{-2\pi i}$, giving (3.14).

---

the branch cuts. On the complex τ_2 plane, this does not lead to any confusion because the choice is always implicitly 'straight upwards' as in the figure. However when we write our correlators in terms of conformal cross ratios it is not obvious where to place the branch cuts on the $z, \bar{z}$ planes. For this reason we will always give the contour description and avoid $i\epsilon$'s entirely in our calculations.

Free 2-point function

A free massless scalar in d dimensions has $\Delta = d/2 - 1$. In even dimensions, this is an integer, so there are no branch cuts in the Lorentzian 2-point function. It follows that $\langle[\phi(x), \phi(y)]\rangle = 0$ at timelike separation. Standard free field methods confirm that the commutator in even dimensions is supported only on the lightcone, $(x - y)^2 = 0$.

3.5 CFT 4-point functions

We now specialize to 4-point functions in a conformal field theory. Take the operators O_1, O_3 and O_4 to be fixed and spacelike separated at $\tau = 0$, while O_2 is inserted at an arbitrary time:

$$x_1 = (0, \ldots, 0), \quad x_2 = (\tau_2, y_2, 0, \ldots, 0), \quad x_3 = (0, 1, \ldots, 0), \quad x_4 = (0, \infty, 0, \ldots, 0) \ , \tag{3.15}$$

with

$$0 < y_2 < \frac{1}{2} \ . \tag{3.16}$$

This is similar to (3.6) but with O_4 moved to infinity.[5] Only one of the operators is at $t \neq 0$, so the others are all spacelike separated. The conformal cross ratios are defined by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ . \tag{3.17}$$

Another convenient notation is

$$u = z\bar{z}, \qquad v = (1 - z)(1 - \bar{z}) \ , \tag{3.18}$$

which for (3.15) becomes[6]

$$z = y_2 + i\tau_2, \qquad \bar{z} = y_2 - i\tau_2 \ . \tag{3.19}$$

In Euclidean signature, τ_2 is real and $\bar{z} = z^*$. In Lorentzian signature, $z = y_2 - t_2$ and $\bar{z} = y_2 + t_2$ are independent real numbers.

---

[5] $O(\infty) \equiv \lim_{y\to\infty} y^{2\Delta_O} O(y)$.

[6] (3.18) is invariant under $z \leftrightarrow \bar{z}$, but we will always choose the solutions of the quadratic equation corresponding to (3.19), so this distinguishes z and $\bar{z}$.

The Euclidean correlator $G(z, z^*)$ has the short-distance singularities

$$G(z, z^*) \sim (zz^*)^{-\frac{1}{2}(\Delta_1+\Delta_2)} \quad \text{as} \quad z \to 0 \tag{3.20}$$

and

$$G(z, z^*) \sim ((1-z)(1-z^*))^{-\frac{1}{2}(\Delta_2+\Delta_3)} \quad \text{as} \quad z \to 1 \; . \tag{3.21}$$

The various Lorentzian correlators are computed by analytic continuation $\tau_2 \to it_2$. Denote by $G(z, \bar{z})$ the time-ordered correlator, defined by analytic continuation along the contour (a) in (3.8). Then for real z and $\bar{z}$, and O_2 in the future lightcone of both O_1 and O_3, the contours in (3.8) correspond to the functions

$$\begin{aligned}
&(a) \qquad && G(z, \bar{z}) = \langle O_2 O_1 O_3 O_4 \rangle \\
&(b) \qquad && G(z, \bar{z})|_{(\bar{z}-1)\to e^{-2\pi i}(\bar{z}-1)} = \langle O_3 O_2 O_1 O_4 \rangle \\
&(c) \qquad && G(z, \bar{z})|_{z\to e^{-2\pi i}z} = \langle O_1 O_2 O_3 O_4 \rangle \\
&(d) \qquad && G(z, \bar{z})|_{z\to e^{-2\pi i}z, (\bar{z}-\bar{z}_0)\to e^{-2\pi i}(\bar{z}-\bar{z}_0)} = \langle O_1 O_3 O_2 O_4 \rangle \; .
\end{aligned} \tag{3.22}$$

These follow from the fact that the first singularity above the real axis in (3.8) is $z = 0$, and the second is $\bar{z} = 1$. The subscripts indicate how to go around these singularities. In the last line, $\bar{z}_0$ is defined to be the singularity of $G(ze^{-2\pi i}, \bar{z})$ as a function $\bar{z}$:

$$G(ze^{-2\pi i}, \bar{z}) \to \infty \qquad \text{as} \qquad \bar{z} \to \bar{z}_0 \; , \tag{3.23}$$

coming from O_3, depicted in (3.10). According to the reconstruction theorems, it must lie on the real axis, $\mathrm{Im}\, \bar{z}_0 = 0$ (so that the singularity in the τ_2 plane lies on the imaginary axis). Comparing contours (c) and (d), the 4-point function is causal if and only if

$$\mathrm{Re}\, \bar{z}_0 \geq 1 \; . \tag{3.24}$$

4 The Lorentzian OPE

In this section we review the Euclidean OPE in d-dimensional CFT, derive some consequences of reflection positivity, and discuss to what extent the OPE can be applied in Lorentzian correlators. For the simplest case where only one operator is timelike separated from the others, we show that there is a convergent OPE channel, and use