9 Symmetries of AdS$_3$

This section consists entirely of exercises. If you are not doing the exercises, then read through them anyway, since this material will be used later in the course. The main goal of this section is derive the famous result of Brown and Henneaux on the central charge of AdS$_3$. This was done in the 80s, using slightly different techniques from what we'll use here, and later came to play an important role in AdS/CFT, as we'll see later.

9.1 Exercise: Metric of AdS$_3$

Anti-de Sitter space is a constant-negative-curvature spacetime. It is the maximally symmetric solution of Einstein's equation with a negative cosmological constant. AdS$_D$ can be realized as a hyperboloid embedded in a $D+1$-dimensional geometry. In this section we will talk about AdS$_3$, which is the hyperboloid

$$X_A X^A = -\ell^2 \tag{9.1}$$

where $A = 0, 1, 2, 3$ is an index in the space Minkowski$_2\times$ Minkowski$_2$, with metric

$$H_{AB} dX^A dX^B = -dX_0^2 + dX_1^2 + dX_2^2 - dX_3^2 \ . \tag{9.2}$$

To find intrinsic coordinates on AdS$_3$, we just need to solve (9.1). One way to solve this equation is by

$$X_0 = \ell \cosh\rho \cos t, \quad X_1 = \ell \sinh\rho \sin\phi, \quad X_2 = \ell \sinh\rho\cos\phi, \quad X_3 = \ell\cosh\rho \sin t \ . \tag{9.3}$$

(a) Check that this solves (9.1), and use (9.2) to find the induced metric on the hyperboloid.

Answer:

$$ds^2 = \ell^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2) \ . \tag{9.4}$$

These are global coordinates on AdS$_3$. Although on the hyperboloid (9.1) we can see from (9.3) that t is a periodic coordinate, when we say 'AdS$_3$' we will always mean the space in which t is 'unwrapped', $t \in (-\infty, \infty)$ (the *universal covering space* of the hyperboloid).

(b) Find the cosmological constant in terms of the AdS radius ℓ.

9.2 Exercise: Isometries

AdS_3 inherits the isometries of the embedding space that preserve the hyperboloid. (For the same reason that the isometries of S^2 are inherited from rotations in R^3.) The group of rotations+boosts in a 4d geometry with signature $(+,+,-,-)$ is $SO(2,2)$, so we expect this to be the isometry group of AdS_3. In this problem we'll confirm this.

(a) As an example, consider the boost vector

$$V = X^1 \partial_{X^0} + X^0 \partial_{X^1} \tag{9.5}$$

in the embedding space (9.2). This preserves the hyperboloid, since under $X^A \to X^A - V^A$, the lhs side of (9.1) is unchanged to linear order (check this).

Write V as an isometry of AdS_3, in the coordinates (9.4). To do this, first define the projection tensor

$$P^A_\mu = \frac{\partial X^A}{\partial x^\mu} \tag{9.6}$$

where x^μ are the coordinates of AdS_3. This can be used to convert the 4-vector V^A into a tensor living on the hyperboloid,

$$\chi_\mu = P^A_\mu V_A \ . \tag{9.7}$$

Find χ^μ, and check that it is a Killing vector of the metric (9.4).

This same procedure can be used to find all of the Killing vectors of AdS, but I will spare you the trouble. The answer, in a convenient basis, is

$$\begin{aligned}
\zeta_{-1} &= \tfrac{1}{2}\left[\tanh(\rho)e^{-i(t+\phi)}\partial_t + \coth(\rho)e^{-i(t+\phi)}\partial_\phi + ie^{-i(t+\phi)}\partial_\rho\right] \\
\zeta_0 &= \tfrac{1}{2}(\partial_t + \partial_\phi) \\
\zeta_1 &= \tfrac{1}{2}\left[\tanh(\rho)e^{i(t+\phi)}\partial_t + \coth(\rho)e^{i(t+\phi)}\partial_\phi - ie^{i(t+\phi)}\partial_\rho\right] \\
\bar{\zeta}_{-1} &= \tfrac{1}{2}\left[\tanh(\rho)e^{-i(t-\phi)}\partial_t - \coth(\rho)e^{-i(t-\phi)}\partial_\phi + ie^{-i(t-\phi)}\partial_\rho\right] \\
\bar{\zeta}_0 &= \tfrac{1}{2}(\partial_t - \partial_\phi) \\
\bar{\zeta}_1 &= \tfrac{1}{2}\left[\tanh(\rho)e^{i(t-\phi)}\partial_t - \coth(\rho)e^{i(t+\phi)}\partial_\phi - ie^{i(t+\phi)}\partial_\rho\right]
\end{aligned} \tag{9.8}$$

Note that the subscripts here are just labels, not spacetime indices.

(b) Check that the vectors ζ_{-1}, ζ_0, ζ_1 are Killing vectors.

(c) Now check that they obey the $SL(2,R)$ algebra:

$$[L_1, L_{-1}] = 2L_0, \quad [L_1, L_0] = L_1, \quad [L_{-1}, L_0] = -L_{-1} \ . \tag{9.9}$$

That is, the Killing vectors obey this algebra under Lie brackets, with an additional i, for example

$$i\{\zeta_1, \zeta_{-1}\}_{LB} = 2\zeta_0 \ , \qquad etc. \tag{9.10}$$

The barred zetas in (9.8) commute with the unbarred zetas, and form another $SL(2,R)$ algebra. Therefore the isometries of AdS_3 form the algebra

$$SL(2,R)_L \times SL(2,R)_R. \tag{9.11}$$

The subscripts mean 'left' and 'right', since the ζ's involve only the 'left-moving' combination $t+\phi$ and the $\bar{\zeta}$'s involve the 'right-moving' combination $t-\phi$.

Note that as a Lie algebra, of $SO(2,2) = SL(2,R) \times SL(2,R)$. This is a special feature of AdS_3. In general the AdS_D isometry group is $SO(D-1,2)$, which does not split into two factors.

9.3 Exercise: Conserved charges

(a) Do the coordinate change

$$t^{\pm} = t \pm \phi \ , \qquad \rho = \log(2r) \ , \tag{9.12}$$

and expand the metric (9.4) at large r. Show that to leading order

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} - r^2 dt^+ dt^- \right) \ . \tag{9.13}$$

These are called *Poincaré coordinates*, and in fact this metric is an exact solution of Einstein's equation – it covers a subregion of AdS_3 called the Poincaré patch.

A spacetime is called *asymptotically AdS* if it approaches (9.13) as $r \to \infty$.[53]

(b) Consider the asymptotically AdS spacetime

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} - r^2 dt^+ dt^- \right) + h_{++}(dt^+)^2 + h_{--}(dt^-)^2 + 2h_{+-}dt^+dt^- \ , \tag{9.14}$$

where the h's are arbitrary functions of t^+ and t^- but independent of r. We will compute the boundary stress tensor (Brown-York tensor) and use it to define the energy and other conserved charges in AdS_3.

The boundary stress tensor is defined as the variation of the on-shell action

$$T^{ij} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{on-shell}}{\delta \gamma_{ij}} \tag{9.15}$$

where the action is

$$S[g] = \frac{1}{16\pi} \int_M \sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi} \int_{\partial M} \sqrt{-\gamma} K + \frac{a}{8\pi} \int_{\partial M} \sqrt{-\gamma} \ . \tag{9.16}$$

For the bulk term and Gibbons-Hawking term, we can use our formulae from flat space given in previous lectures. The last term is a counterterm, which takes the place of the

[53] Specifically, the subleading components of the metric must have a certain fall-off at large r. These conditions are basically chosen so that the Hamiltonian can be defined.

'background subtraction' we did in flat space. This leads to

$$T^{ij} = \frac{1}{8\pi}\left[K^{ij} - K\gamma^{ij} + \tilde{a}\gamma^{ij}\right] \ . \tag{9.17}$$

Choose the counterterm coefficient $\tilde{a}$ so that T_{ij} is finite as the cutoff surface $r_0 \to \infty$. Compute T_{++}, T_{--}, and T_{+-} to first order in the perturbation h_{ij} in the limit $r_0 \to \infty$.

Reference: Balasubramanian and Kraus, hep-th/9902121.

(c) Compute the energy of the spacetime (9.14). It is defined as

$$E = \frac{1}{\ell}\int_0^{2\pi} d\phi\sqrt{\sigma}u^i T_{ij}\zeta^j \tag{9.18}$$

where u^i is the timelike normal to a fixed-t slice, and $\zeta = \partial_t$. (The overall $1/\ell$ is a convention, necessary due to the fact we are using dimensionless coordinates.)

(d) An example of an asymptotically AdS spacetime is the BTZ black hole,

$$ds^2 = \ell^2\left[-(r^2 - 8M)dt^2 + \frac{dr^2}{r^2 - 8M} + r^2 d\phi^2\right] \ . \tag{9.19}$$

Check that for this spacetime

$$E = M \ . \tag{9.20}$$

To use your results of the previous problem in this calculation you must first change coordinates to put it in the form (9.14). In particular you will need to redefine $r' = r'(r)$ to eliminate the perturbation to g_{rr}.

(e) Compute the energy of global AdS, by keeping the subleading terms in the coordinate transformation (9.12) and plugging them into your formula for the stress tensor. (*Hint:* the answer is negative. That's OK, this is just a choice of zero for energy.)

Comment: We've focused on the energy, but we could compute conserved charges corresponding to all the other Killing vectors in exactly the same way.