8 Symmetries and the Hamiltonian

Throughout the discussion of black hole thermodynamics, we have always assumed energy $= M$. Now we will introduce the Hamiltonian formulation of GR and show how to define conserved charges associated to spacetime symmetries. The energy is a special case, associated to time-translation symmetry. There are quicker ways to reach the conclusion energy $= M$ (see Carroll's book), but we will take the more careful route because it's useful later.

8.1 Parameterized Systems

[*References:* The original paper is very nice and still worth reading, especially sections 1-3: "The Dynamics of General Relativity" by Arnowitt, Deser, Misner (ADM), 1962 (but available on arXiv at gr-qc/0405109). See also appendix E of Wald's textbook, and for full detail see Poisson's *Relativist's Toolkit* chapter 4.]

Time plays a special role in the canonical formulation of quantum mechanics, and in the Hamiltonian approach to classical mechanics, since it is the independent variable. In GR, time t is just an arbitrary parameter, and the dynamics are reparameterization-invariant under $t \to t'(t)$, since this is just a special case of diffeomorphisms. To see how this fits into Hamiltonian mechanics we first consider a simple analog in quantum mechanics.

Suppose we have a system with a single degree of freedom $q(t)$ with conjugate momentum p, and action

$$I = \int dt L. \tag{8.1}$$

The Hamiltonian is the Legendre transform

$$H(p, q) = p\dot{q} - L(q, \dot{q})|_{p=\partial L/\partial \dot{q}} \ . \tag{8.2}$$

Hamilton's equations of motion are

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} \ , \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p} \ . \tag{8.3}$$

The independent variable t is special. It labels the dynamics but does not participate as a degree of freedom. In GR, time is just an arbitrary label – it is not special, and the theory is invariant under time reparameterizations. To mimic this in our simple system with 1 dof, we will introduce a fake time-reparameterizations symmetry. To do this we label the dynamics by an arbitrary parameter τ, and introduce a physical 'clock' variable T, treated as a dynamical degree of freedom. So instead consider the system of variables and conjugate momenta

$$q(\tau), \quad p(\tau), \quad T(\tau), \quad \Pi(\tau) \tag{8.4}$$

where Π is the momentum conjugate to T. This is equivalent to the original 1 dof if we use the 'parameterized' action

$$I' = \int d\tau (pq' + \Pi T' - NR) \ , \qquad R \equiv \Pi + H(p,q) \ , \tag{8.5}$$

where prime $= d/d\tau$. Here $N(\tau)$ is a Lagrange multiplier, which enforces the 'constraint equation'

$$\Pi + H(p,q) = 0 \ . \tag{8.6}$$

The action (8.5) is reparameterization invariant under $\bar{\tau} = \bar{\tau}(\tau)$, since after all τ is just a label that we invented. The Hamiltonian of the enlarged system is simply

$$H' = N(\Pi + H(p,q)) \ , \tag{8.7}$$

so it vanishes on-shell due to the constraint equation!

To recap: we introduced time-covariance by adding a fake degree of freedom, and imposing a constraint. The resulting Hamiltonian vanishes on-shell, because it generates τ-translations, which is just part of the reparameterization symmetry.

To reverse the procedure, *i.e.,* to go from the parameterized action back to the ordinary action with 1 dof, we plug in the constraint

$$I' = \int d\tau \, [pq' - H(p,q)T'] \tag{8.8}$$

and then rewrite the dynamics in terms of the clock variable:

$$I' = \int dT \, [p\dot{q} - H(p,q)] \tag{8.9}$$

where dot $= d/dT$. So we see that T is just the original physical time t.

The equation of motion for T is

$$T' = N \frac{\partial}{\partial T}(\Pi + H) \tag{8.10}$$

But both T' and N are unspecified by the dynamics. For example we are free to pick the 'gauge condition' $T = \tau$, which corresponds to some particular choice of $N(\tau)$.

8.2 The ADM Hamiltonian

GR is already a 'parameterized system:' the t coordinate is like our τ coordinate above, and we will see that the Hamiltonian is very much like (8.7).

The canonical variables are

$$h_{ij}(\vec{x}, t), \qquad \pi_{ij}(\vec{x}, t) \tag{8.11}$$

where h_{ij} are the space components of the metric, and π_{ij} are their canonical conjugates. The full spacetime metric is parameterized as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ . \tag{8.12}$$

$N = \sqrt{-1/g^{tt}}$ is called the 'lapse' and $N^i = N^2 g^{ti}$ is the 'shift'. These are Lagrange multipliers, just like N in our discussion above. They are not fixed by the dynamics, but a choice of parameterization. In other words, *any geometry can be sliced into 'time' and 'space' in a such a way that N and N^i can be set to any functions you like.* They are called the lapse and shift because they correspond to our choice of how our coordinates on a time-slice of fixed $t = t_0$ are related to the coordinates on a time-slice of fixed $t = t_0 + \delta t$. The flow vector, which tells you the arrow of time from one slice to the next, is[43]

$$\zeta^\mu = N u^\mu + N^\mu \ . \tag{8.13}$$

In the coodinates (8.12), $\zeta = \partial_t$, but we will treat N and N^a as arbitrary parameters.

The action of GR, discussed above but now in Lorentzian signature, is

$$I = \frac{1}{16\pi}\int_M d^4x\,\sqrt{-g}R - \frac{1}{8\pi}\int_{\partial M} d^3x\,\sqrt{-\gamma}(K - K_0) \tag{8.14}$$

where K_0 is the subtraction term (the extrinsic curvature of the boundary embedded in flat spacetime). Recall that the on-shell variation is

$$\delta I_{on-shell} = \tfrac{1}{2}\int_{\partial M} d^3x\sqrt{-\gamma}T^{ij}\delta g_{ij}, \tag{8.15}$$

where the 'boundary stress tensor' (aka Brown-York stress tensor) is

$$T^{ij} = \frac{1}{8\pi}\left(K^{ij} - \gamma^{ij}K\right) - \text{background subtraction} \ . \tag{8.16}$$

After quite a bit of work[44], the full off-shell action (8.14) can be written

$$I = \int_M d^4x\left[\pi^{ij}\dot{h}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i\right] - \int_{\partial M} d^3x\,\sqrt{\sigma}u^\mu T_{\mu\nu}\zeta^\mu \ , \tag{8.17}$$

From here we can read off the Hamiltonian[45]

$$H[\zeta] = \int_\Sigma d^3x\,(N\mathcal{H} + N^i\mathcal{H}_i) + \int_{\partial\Sigma} d^2x\,\sqrt{\sigma}u^\mu T_{\mu\nu}\zeta^\nu \ , \tag{8.18}$$

which is an integral over a spatial slice Σ.

Now to explain all these terms: $\mathcal{H}$ and $\mathcal{H}_i$ are called the Hamiltonian and momentum constraints, which are essentially the G_{00} and G_{0i} components of the Einstein equations (see Wald for explicit formulae). These components of the equations of motion involve only 1st time derivatives. They are called 'constraints' because if we think of GR as an initial value problem – specify initial data, then evolve in time according to the

[43] Here $N^\mu = h^\mu_a N^a$, where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, *i.e.,* h^μ_a is the projector onto a spatial slice.

[44] See Brown and York, "Quasilocal energy and conserved charges derived from the gravitational action, " 1993, and also Poisson's *Relativist's Toolkit*, Chapter 4.

[45] The bulk term is called the 'ADM Hamiltonian'. As far as I know, the boundary terms were first derived by Brown and York, and by Hawking and Horowitz.

dynamical equations – these are constraints on the allowed initial data $h_{ij}, \dot{h}_{ij}$ at $t = 0$. This is in contrast to other, dynamical equations of motion, which tell you how that data evolves in time.[46] Finally u^μ is the timelike unit normal on the boundary, with $u^2 = -1$.

A few remarks about our final answer (8.18):

- The bulk term vanishes on-shell due to the constraint equations. The boundary term does not vanish in general. This is related to the fact that diffeomorphisms acting on the boundary are 'real' dynamics, whereas diffeomorphisms away from the boundary are just redundancies.

- We have written the Hamiltonian as a functional of the lapse and shift, since the dynamics leaves ζ unspecified. This corresponds to a choice of time evolution. That is, the Dirac bracket[47] of the Hamiltonian with any function X of the canonical variables is

$$\{H[\zeta],\ X\} = \mathcal{L}_\zeta X \ . \tag{8.19}$$

  If we choose, for example, $\zeta^\mu = (1, 0)$, then this Hamiltonian generates time evolution in the t-direction.

- The on-shell Hamiltonian looks just like the Hamiltonian of a 3-dimensional theory living on the boundary with a 3-dimensional stress tensor $T_{\mu\nu}$. We will see that at least in AdS this is actually literally the case.

Exercise: Constraints in electrodynamics

Difficulty level: medium

Derive the Hamiltonian of electrodynamics. Start from the action $I = -\frac{1}{4}\int d^4x F_{\mu\nu}^2$,

[46] In electrodynamics, the action involves only first derivatives of A_t, so this is a Lagrange multiplier like the lapse in GR. The Hamiltonian of electrodynamics has a term $A_t C$ where $C = \nabla \cdot E - \rho_{matter}$ is the Gauss constraint.

[47] The Dirac bracket is the Poisson bracket, but accounting for gauge symmetries which modify the bracket acting on physical fields. The Dirac bracket is what becomes a commutator in the quantum theory.

identify the canonical coordinates and conjugate momenta, and rewrite it like we did for gravity in (8.17). Identify the Lagrange multiplier(s) and constraint(s).

In gravity, we gave a physical interpretation of the lapse and shift Lagrange multipliers as a choice of foliation of spacetime. What is the analogous interpretation of A_t in electrodynamics? (It might be useful to couple to a matter field to answer this.)

Reference: Appendix E of Wald. But write your answers in terms of the vector potential, not $\vec{E}$ and $\vec{B}$.

8.3 Energy

As usual, the numerical value of the Hamiltonian, evaluated on a solution, is the energy. In GR we must specify a lapse and shift to define the Hamiltonian. The energy is associated to time translations, so we identify the energy as the Hamiltonian with $N \to 1$ and $N^a \to 0$ at the boundary. For this choice, the surface deformation vector $\zeta_{(t)}$ has components $\zeta^\mu_{(t)} = (1, 0, 0, 0)$, so

$$E \equiv H[\zeta_{(t)}]|_{on-shell} = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^i T_{it} \ . \tag{8.20}$$

This is the usual (covariantized) expression for the energy in terms of the stress tensor.[48]

We must impose boundary conditions to ensure that energy is conserved. It can be shown that the $G_{r\mu}$ components of the Einstein equations[49] are

$$\nabla_i T^{ij} = -n_\alpha T^{\alpha j}_{matter} \ , \tag{8.21}$$

where n is the spacelike unit normal to the boundary. Therefore, if we impose the boundary condition that matter fields go to zero fast enough as $r \to \infty$, then the

[48] This equation agrees with other definitions of energy you may have seen, like the Komar formula, whenever those definitions apply.

[49] *i.e.,* the 'constraints' in a radial slicing of the spacetime, which contain only first order r-derivatives.

boundary stress tensor is conserved,

$$\nabla_i T^{ij} = 0 \ . \tag{8.22}$$

If in addition

$$\nabla_{(i}\zeta_{j)} = 0 \qquad \text{as} \qquad r \to \infty \tag{8.23}$$

then the energy current $j^i = T^i{}_j \zeta^j$ is conserved, $\nabla_i j^i = 0$. In this case the energy is independent of what slice Σ we choose to evaluate (8.20):

$$E(\Sigma) - E(\Sigma') = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^i T_{it} - \int_{\partial\Sigma'} d^2x \sqrt{\sigma} u^i T_{it} = \int d^3x \sqrt{-\gamma} \nabla_i \left(T^{ij} \zeta_j \right) = 0 \ . \tag{8.24}$$

The equation (8.23) is the Killing equation, so the conclusion is that energy is conserved as long as (i) matter fields fall off fast enough near infinity, and (ii) $\zeta = \partial_t$ is an *asymptotic Killing vector*.

What about matter?
The expression (8.20) includes the contribution from matter. The constraints ensure that the metric at infinity knows about any matter localized in the interior: the matter backreacts on the metric, and therefore contributes at infinity. This is just like the Gauss law in E&M.

8.4 Other conserved charges

Other asymptotic Killing vectors will similarly lead to conserved quantities. For example, if $\zeta = \partial_\phi$ satisfies (8.23), then we can define the conserved charge

$$J = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^i T_{i\phi} \ . \tag{8.25}$$

This is in fact the angular momentum, and agrees with all the usual formulae for computing the angular momentum of a spacetime.

We could also define boost charges, and get the full Poincare group. This requires some modifications, since in this discussion ζ_i was a vector within the fixed ∂M, whereas boosts act on ∂M. The results are similar.

8.5 Asymptotic Symmetry Group

We have seen that the bulk Hamiltonian vanishes, but there are boundary terms that compute conserved charges. Now I will try to explain physically what is going on here.

Local diffs are fake. Global diffs are real.
GR is locally diff invariant, but it is not invariant under diffs that reach the boundary. To see this from the action, just vary it under a general diff ζ. The Lie derivative for a density is

$$\delta_\zeta(\sqrt{g}f) \equiv \mathcal{L}_\zeta(\sqrt{g}f) = \nabla_\mu(f\zeta^\mu) \ . \tag{8.26}$$

Applying this the Lagrangian density of GR we see that it is only diff-invariant up to a boundary term,

$$\int_M \delta_\zeta(\sqrt{g}\mathcal{L}) = \int_{\partial M} dA^\mu \zeta_\mu \mathcal{L} \ . \tag{8.27}$$

This is important, so I'll rephrase: General relativity is invariant under local diffeomorphisms. These are like gauge symmetries: fake symmetries, redundancies, that do not change the physics and are just a convenient human invention to describe massless particles. However it is *not* invariant under diffeomorphisms that reach the boundary. The coordinates as $r \to \infty$ are actually important and meaningful, like the coordinates in a non-gravitational theory. A time reparameterization with compact support, *i.e.,* $t \to t'(t,x)$ such that $t' \to t$ as $r \to \infty$, is a local diff and involves no physics. A global time shift $t \to t+1$ acts at infinity and is true time evolution.

The bulk terms in the Hamiltonian, *i.e.,* the constraints, correspond to local diffs, and the boundary terms correspond to diffs that reach the boundary. That is why the bulk term vanishes on shell and the boundary term does not.

Certain diffs that reach infinity are actual symmetries. By 'actual' symmetries, I mean symmetries that act on the space of states in the theory: they take one state to a *distinct but related* state with similar properties, as opposed to gauge symmetries which physically do nothing.

Asymptotic symmetries in $U(1)$ gauge theory

The precise version of all these statements is the formalism of *asymptotic symmetries*. The definition of the asymptotic symmetry group is the group of symmetry transformations modded out by trivial symmetries,

$$ASG = \frac{\text{symmetries}}{\text{trivial symmetries}} \ . \tag{8.28}$$

The definition of a 'trivial symmetry' is one whose associated conserved charge vanishes.

Let's consider electromagnetism as an example. The action $I = -\frac{1}{4}\int d^4x(F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu_{matter})$ is invariant under an infinite number of transformations,

$$\delta A_\mu = \partial_\mu \Lambda(x) \ , \qquad \delta\phi = i\Lambda(x)\phi \ , \tag{8.29}$$

where the second term indicates the usual phase rotation on charged matter. These are gauge symmetries. A local gauge symmetry, ie a transformation for which $\Lambda(x)$ has compact support, does not have any conserved charge associated to it. In fact despite this infinite number of symmetries we know electromagnetism has only one conserved quantity, the total charge

$$Q \sim \int_\Sigma d^3x J^0_{matter} \sim \int_{\partial\Sigma} d^2x \, F_{tr} \ . \tag{8.30}$$

This is the conserved charge associated to the global $U(1)$ rotation – it exists and is conserved even in the un-gauged theory. Thus the global rotation is physical, while local phase rotations are just redundancies.

The definition (8.28) of the asypmtotic symmetry group is the group of all transformations, mod gauge transformations with zero associated charge. Therefore in electromagnetism,

$$ASG = U(1)_{global} \ . \tag{8.31}$$

Asymptotic symmetries in gravity

We will not go into depth on the ASG in gravity right now, but I will just mention some facts. The ASG in gravity is generated by the conserved charges, which we argued above are the charges associated to some special vector fields, including those for which $\nabla_{(i}\zeta_{j)} \to 0$ at infinity. In asymptotically flat spacetimes, ie spacetimes approaching Minkowski space fast enough as $r \to \infty$, these are simply the Killing vectors of Minkowski space. Thus the asymptotic symmetry group of asymptotically flat spacetimes is the Poincare group.[50]

This notion is important, because general spacetimes have no isometries, and therefore no local conserved charges. (For example, there is no 'energy' conserved along the geodesic of a probe particle.) Asymptotic symmetries allow us to define global conserved quantities in this situation.

The Poincare algebra in 4D has 10 generators: 4 translations P_μ and 6 Lorentz generators $M_{\mu\nu}$. The generators obey the Poincare Lie algebra

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 && (8.32)\\
\frac{1}{i}[M_{\mu\nu}, P_\rho] &= \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu && (8.33)\\
\frac{1}{i}[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\ . && (8.34)
\end{aligned}$$

If we just label the generators as V^A for $A = 1\dots 10$, then this is just a Lie algebra

$$i[V^A, V^B] = f^{AB}_{\ \ C}V^C \tag{8.35}$$

with some structure constants $f^{AB}_{\ \ C}$. Each of these generators is associated to a Killing vector of Minkowski space:

$$V^A \leftrightarrow \zeta^{(A)\mu}, \qquad A = 1\dots 10\ . \tag{8.36}$$

For example $P^\mu \leftrightarrow \partial_\mu$, $M_{tx} \leftrightarrow t\partial_x + x\partial_t$, etc. The Killing vectors obey the same algebra, under the Lie bracket:

$$[\zeta^A\ ,\ \zeta^B]^\mu_{LB} \equiv \zeta^{A\nu}\partial_\nu\zeta^{B\mu} - \zeta^{B\nu}\partial_\nu\zeta^{A\mu} = f^{AB}_{\ \ C}\zeta^{C\mu}\ . \tag{8.37}$$

(Here A is a label of which vector, and μ is a spacetime index.)

[50] This is true at spacelike infinity. The story at null infinity is much more subtle since, in non-static spacetimes, gravitational radiation reaches null infinity and distorts the asymptotics. This leads to what is called the BMS group, which is an active area of research.

Recall that conserved charges generate the action of the diffeomorphism under Dirac brackets. That is, the charge

$$Q^A = H[\zeta^A] \tag{8.38}$$

generates

$$\{Q^A, X\}_{DB} = \mathcal{L}_{\zeta^A} X \ . \tag{8.39}$$

For this to be consistent with the algebra, the charges themselves must obey the same algebra:

$$\{Q^A \ , \ Q^B\}_{DB} = f^{AB}{}_C Q^C \ . \tag{8.40}$$

In other words,

$$\{H[\zeta], H[\chi]\}_{DB} = H\big[[\zeta,\chi]_{LB}\big] + \text{constant} \ , \tag{8.41}$$

where we have allowed a constant 'central charge' term in the algebra of charges, since this would still be consistent with the action of the generators on X (and actually does appear in important examples).

Sometimes the ASG leads to surprises. A famous example is in anti-de Sitter space. The isometry group of AdS_D is $SO(D-1,2)$. So a natural guess is that the asymptotic symmetry group of asymptotically-AdS spacetimes is also $SO(D-1,2)$. This is true for $D > 3$ but wrong in $D = 3$, as shown by Brown and Henneaux. We will talk about this more later.

8.6 Example: conserved charges of a rotating body

The linearized solution of GR that carries both energy and angular momentum is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right)(dr^2 + r^2 d\Omega^2) - \frac{4j\sin^2\theta}{r} dt d\phi \ . \tag{8.42}$$

This is, for example, the metric far away from a Kerr black hole, or a rotating planet.

We will compute the energy and angular momentum using the on-shell Hamiltonian (8.18). Here it is again, after enforcing the constraints:

$$H[\zeta] = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^i T_{ij} \zeta^j \ . \tag{8.43}$$

The energy is associated to $\zeta = \partial_t$ and the angular momentum to $\zeta = \partial_\phi$.

Kinematics

We want to compute T^{ij}. This is a tensor living on ∂M, which is the surface $r = r_0$. To define tensors on ∂M, we first compute the unit normal to the ∂M,

$$n_\mu dx^\mu = \sqrt{1 + \frac{2M}{r}} dr \ . \tag{8.44}$$

The full metric can be split into the normal and tangential parts as

$$g_{\mu\nu} = \gamma_{\mu\nu} + n_\mu n_\nu \ . \tag{8.45}$$

γ^μ_ν projects onto the boundary, since $n_\mu \gamma^\mu_\nu = 0$. The components γ^μ_i for $\mu = t, r, \theta, \phi$ and $i = t, \theta, \phi$ can be used to turn spacetime tensors into boundary tensors, and vice-versa:

$$V_i \equiv \gamma_i^\mu V_\mu \ . \tag{8.46}$$

The induced metric on ∂M is

$$\gamma_{ij} dx^i dx^j = -\left(1 - \frac{2M}{r_0}\right) dt^2 + r_0^2 \left(1 + \frac{2M}{r_0}\right)(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4j}{r_0} \sin^2\theta d\phi dt \ . \tag{8.47}$$

The timelike unit normal to a constant-t hypersurface is

$$u_\mu dx^\mu = (-1 + \frac{M}{r} + O(1/r^2)) dt \ . \tag{8.48}$$

(This could be used to define the induced metric from $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and corresponding projector but we won't need these to compute the charges.) Projecting the timelike normal onto the boundary doesn't change anything, we still have

$$u_i dx^i = (-1 + \frac{M}{r_0} + O(1/r_0^2)) dt \ , \tag{8.49}$$

where remember i runs over the boundary directions $x^i = (t, \theta, \phi)$.

Finally, we need the volume element of the boundary at fixed time. The induced metric on $\partial\Sigma$ is

$$\sigma_{AB}dx^A dx^B = r_0^2(1+\frac{2M}{r_0})(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{8.50}$$

with volume element

$$\sqrt{\sigma} = r_0^2\left(1+\frac{2M}{r_0}\right)\sin\theta \ . \tag{8.51}$$

Stress tensor

The extrinsic curvature of ∂M is

$$K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)} \ . \tag{8.52}$$

As a boundary tensor,

$$K_{ij} = h_i^\mu h_j^\nu K_{\mu\nu} \ . \tag{8.53}$$

The trace of K is the same whether we use $K_{\mu\nu}$ or K_{ij} (check this!). It is

$$K = -\frac{2}{r_0} - \frac{3M}{r_0^2} + O(r_0^{-3}) \ . \tag{8.54}$$

Now we compute the stress tensor from its definition (ignoring the background subtraction for now), $T_{ij} = K_{ij} - \gamma_{ij}K$. (I've rescaled it by 8π to unclutter notation, but will put the 8π back in the Hamiltonian below.) It has components

$$T_{tt} = -\frac{2}{r_0} + \frac{8M}{r_0^2}, \quad T_{t\phi} = -\frac{5j\sin^2\theta}{r_0^2}, \quad T_{\theta\theta} = r_0 + M, \quad T_{\phi\phi} = \sin^2\theta(r_0+M) \tag{8.55}$$

plus higher order terms $O(M^2/r_0^2)$. (In this equation we are still ignoring the background subtraction, we will deal with that below.)

Energy

The energy is the on-shell Hamiltonian for $\zeta = \partial_t$. Putting it all together, we have so far for the energy

$$E_{unsub} = \frac{1}{8\pi}\int_{\partial M} 2r_0 \sin\theta = -r_0 \ , \tag{8.56}$$

where 'unsub' means we have not dealt with the background subtraction yet.

To do the background subtraction, we repeat the whole calculation on the flat spacetime

$$ds^2_{sub} = -\left(1 - \frac{2M}{r_0}\right) dt^2 + \left(1 + \frac{2M}{r_0}\right) (dr^2 + r^2 d\Omega^2) \ . \tag{8.57}$$

This is a flat spacetime with the same intrinsic geometry on ∂M.[51]

Going through all the steps again, the subtraction term is $E_{sub} = -r_0 - M$. Therefore the final answer is

$$E = M \ , \tag{8.58}$$

as expected.

Angular momentum

The angular momentum is the on-shell Hamiltonian for $\zeta = -\partial_\phi$.[52] There is no background subtraction necessary. We find

$$J = \frac{1}{8\pi} 3j \int_{\partial\Sigma} d\theta d\phi \sin^3\theta = j \ . \tag{8.59}$$

[51]We could include the angular momentum term, but we can shift the time coordinate to make it $O(1/r_0^2)$ and it does not contribute. Put differently, we really only need to embed $\partial\Sigma$ into flat spacetime, not all of ∂M, so this is only important for the energy and we can ignore the angular momentum.

[52]The minus sign here is the standard convention. It is related to the fact that a mode $e^{-i\omega t + im\phi}$ carries energy $E = +\omega$ and angular momentum $J = +m$.