7 Thermodynamics of de Sitter space

Inflation is the idea that the very early universe, at $t \lesssim 10^{-32}s$, is approximately de Sitter spacetime. We will now take a detour to apply our methods to de Sitter space, since in this situation they produce observable effects via the imprint of primordial density perturbations on the CMB.

de Sitter is the maximally symmetric solution of the Einstein equations with a positive cosmological constant. The metric is[37]

$$ds^2 = -dT^2 + \ell^2 \cosh^2(T/\ell)d\Omega_{D-1}^2 \qquad \text{(global coordinates)} . \qquad (7.1)$$

These coordinates, which cover all of de Sitter space, describe a sphere S^{D-1} that is very large at $T \to -\infty$, contracts to a minimum radius ℓ at $T = 0$, then expands as $T \to \infty$. The length scale is set by the cosmological constant, $\Lambda \sim 1/\ell^2$.

To draw the Penrose diagram, we need to make the range of T finite. This can be done by defining

$$\tan(\eta/2) = \tanh(T/2\ell) . \qquad (7.2)$$

In these coordinates, which run over $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the metric is

$$ds^2 = \frac{\ell^2}{\cos^2 \eta} \left(-d\eta^2 + d\Omega_{D-1}^2 \right) . \qquad (7.3)$$

For the Penrose diagram (which captures the causal structure but ignores distances) we can ignore the overall factor, so this gives the diagram in figure 3. Note that de Sitter space has an initial and final conformal boundary. (Although the diagram also appears to have left and right boundaries, these are not really boundaries – at each value of η space is a sphere, so those lines are just the north and south poles of the sphere S^{D-1}.)

Vacuum

As usual, there is no unique vacuum. However we can follow our usual prescription,

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[37]I will call these global coordinates, but in the context of cosmology this is usually called the 'closed slicing.'
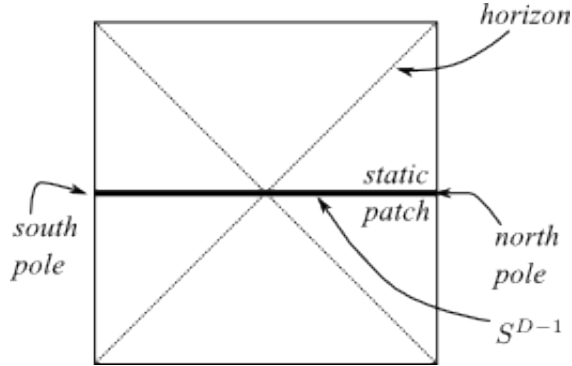
Figure 3: Penrose diagram of de Sitter space.

as we did for Minkowski and Schwarzschild, and define a vacuum which is prepared by a path integral on the Euclidean continuation of de Sitter. Euclidean de Sitter space is just a sphere: send $T \to i\ell\theta$ in (7.1), and you see that $\theta \in [-\pi/2, \pi/2]$ is the polar coordinate on a sphere S^D of radius ℓ. It is shifted by $\pi/2$ from the usual definition, so the surface $T = \theta = 0$ is the equator of the S^D. This equator is the minimal-size spatial section of de Sitter space, an S^{D-1} of radius ℓ. This is the $T = 0$ surface at the middle of the Penrose diagram.

The *Euclidean vacuum*[38] is the state prepared by a path integral on the hemisphere. This path integral prepares a quantum state on the equator, which can then be evolved in Lorentzian time. Here is a cartoon for this path integral:

$$(7.4)$$
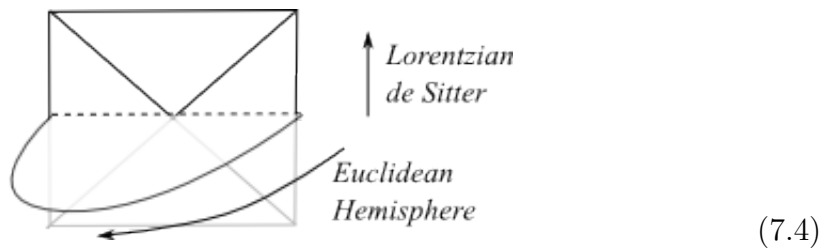
Roughly speaking, 'most' states will eventually end up close to the Euclidean vacuum, since the de Sitter expansion dilutes any excitations.

This vacuum state has in a sense been observed experimentally, since it leaves an imprint in the CMB. I say 'in a sense' because it is not a very fine-grained or direct test of the details of de Sitter space, but it is nonetheless the only good explanation

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[38]Also called the 'Bunch-Davies vacuum' and various other things.

we have for those fluctuations.

7.1   Vacuum correlators

Now we will calculate the Gaussian fluctuations in de Sitter that are seen in the CMB, using path integral methods.[39] We consider a free massless scalar field, in the 'conformal coordinates' (7.3) in $D = 4$. (It turns out that scalar fluctuations of the metric reduce to this problem.) The action is

$$I \sim \int_M \sqrt{-g} \nabla_\mu \phi \nabla^\mu \phi \ . \tag{7.5}$$

We want to compute the wavefunction for this scalar field, in the Euclidean vacuum. This wavefunction is defined to be the transition amplitude

$$\Psi[\phi_0; t_0] = \langle \phi_0 | e^{-iHt_0} | 0 \rangle \ . \tag{7.6}$$

This is a path integral over field configurations on the mixed spacetime (7.4). The past boundary condition is regularity on the Euclidean sphere; the future boundary condition is $\phi(\Omega, t_0) = \phi_0(\Omega)$.

In the WKB approximation the wavefunction is simply the on-shell action of a classical solution satisfying these boundary conditions,

$$\Psi[\phi_0] \sim e^{iI[\phi]} \ . \tag{7.7}$$

In fact, since this is a free field, this expression is exact. The on-shell action is a pure boundary term, since we can integrate by parts in (7.5) then use the equations of motion. Thus

$$I_{on-shell}(t_0) \sim \int_{\Sigma(t_0)} \sqrt{h} \phi \partial_n \phi \tag{7.8}$$

where $\Sigma(t_0)$ is the spatial slice at time t_0, ∂_n is the derivative normal to this slice, and h is the spatial metric.

At late times, the S^{D-1} spatial sphere in (7.3) is very large. When we look back at the

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[39]This is based on Maldacena astro-ph/0210603.

CMB, we are looking at a tiny patch of this sphere. Therefore at late times we can think of the spatial sphere as being essentially flat. That is, we can only see modes with wavelength much smaller than ℓ. This simplifies the calculation of the wavefunction because at late times we can approximate the metric by

$$ds^2 = \frac{\ell^2}{\eta^2}(-d\eta^2 + d\vec{x}^2) \ . \tag{7.9}$$

This is enough to compute the wavefunction, since (7.8) is a pure boundary term at $t = t_0$ ($\eta = \eta_0$).

For a plane wave $\phi_0 = h_k(\eta)e^{i\vec{k}\vec{x}}$, the wave equation at late times (*i.e.*, using the metric (7.9)) is

$$\Box\phi = 0 \qquad \Rightarrow \qquad \partial_\eta^2 h_k(\eta) - \frac{2}{\eta}\partial_\eta h_k(\eta) + \vec{k}^2 h_k(\eta) = 0 \ . \tag{7.10}$$

In order to find the wavefunction, we need to find the classical solution satisfying the boundary conditions stated above. This is

$$\phi = \phi_k^0 \frac{(1 - ik\eta)e^{ik\eta}}{(1 - ik\eta_0)e^{ik\eta_0}} \ . \tag{7.11}$$

This satisfies all of our criteria: it solves the wave equation, and it is equal to ϕ_k^0 at time $\eta = \eta_0$. The last criterion was regularity on the Euclidean sphere – this is what picks the solution going as $e^{+ik\eta}$, rather than the other solution $\sim e^{-ik\eta}$. To see this, note the pole of the Euclidean sphere is $t = i\ell\pi/2$, which corresponds to

$$\eta = 2\tan^{-1}(i\pi/4) = i\infty \ . \tag{7.12}$$

Only the $e^{+ik\eta}$ solution is regular in this limit, so we've picked the correct solution, corresponding to the Euclidean vacuum state. (This is very similar to the path-integral calculation of the groundstate for the harmonic oscillator.) Although ϕ is a real field, (7.11) is complex. This is fine: even though the path integral is over real field configurations, the stationary phase approximation can pick out a complex saddlepoint. It is just a trick to compute the path integral.

Plugging (7.11) into (7.8),

$$iI = i \int \frac{d^3k}{(2\pi)^3} \frac{\ell^2}{2\eta_0^2} \phi_{-k}^0 \partial_\eta \phi_k^0|_{\eta=\eta_0} \sim \int \frac{d^3k}{(2\pi)^3} \frac{\ell^2}{2} \left[i\frac{k^2}{\eta_0} - k^3 + \cdots \right] \phi_{-k}^0 \phi_k^0 \qquad (7.13)$$

This gives the wavefunction (exactly in a free theory) via (7.7). Knowing the wavefunction, we can calculate correlators using $\langle \phi^2 \rangle \sim \int D\phi \phi^2 |\Psi(\phi)|^2$. In detail, accounting for the normalization of the wavefunction:

$$\langle \phi_k \phi_{-k} \rangle = \frac{\int D\phi\, \phi_k \phi_{-k} \exp\left(- \int \frac{d^3p}{(2\pi)^3} \ell^2 p^3 \phi_p \phi_{-p}\right)}{\int D\phi \exp\left(- \int \frac{d^3p}{(2\pi)^3} \ell^2 p^3 \phi_p \phi_{-p}\right)}$$

$$= (2\pi)^3 \frac{1}{2\ell^2} k^{-3} , \qquad (7.14)$$

where we used the Gaussian integral $\int dz z^2 e^{-az^2} / \int dz e^{-az^2} = \frac{1}{2a}$ (and we've dropped an overall momentum-conserving delta function).

The power law in (7.14) is what is meant by the statement that 'inflation predicts a scale-invariant spectrum of primordial scalar perturbations.'

7.2   The Static Patch

A single inertial observer travels on a geodesic which we might as well call the north pole of S^{D-1}. Thus the worldline of an observer is the solid line on the right side of the Penrose diagram. It is clear from the diagram that this observer is in causal contact with only a subregion of de Sitter. This is because the universe is expanding very rapidly, so you cannot communicate with someone beyond a certain critical distance: the cosmological horizon. This is labeled 'horizon' in figure 3. Like Rindler space, the position of the horizon depends on the observer.

The *static patch* is the region of de Sitter in causal contact with an observer sitting at the north pole. This is the analogue of the Rindler patch. The coordinates on the static patch are

$$ds^2 = -(1 - r^2/\ell^2)dt^2 + \frac{dr^2}{1 - r^2/\ell^2} dr^2 + r^2 d\Omega_{D-2}^2 . \qquad (7.15)$$

This looks like a black hole except, the static patch where an observer lives is the *inside*, $r < \ell$, with a cosmological horizon at $r = \ell$.

Temperature

In the Euclidean vacuum, an observer in the static patch will see a temperature.[40] This is the same reason we discussed above for Rindler space and then for the Schwarzschild spacetime in the Hartle-Hawking state. The Euclidean path integral prepares a state entangled between the left and right static patches, and when we trace over the hidden region, the resulting density matrix is thermal. To find the temperature we can apply the imaginary time periodicity trick. Starting from the static patch coordinates, define $r = \ell(1 - \epsilon^2)$ and expand in ϵ to find:

$$ds^2 \approx 2\ell^2(d\epsilon^2 - \frac{\epsilon^2}{\ell^2}dt^2) + \cdots \tag{7.16}$$

This looks like Rindler, so the Euclidean continuation is regular only if $t \sim t + 2\pi i\ell$. Therefore the de Sitter temperature is

$$T_{dS} = \frac{1}{2\pi\ell} \ . \tag{7.17}$$

This is the temperature that you will read on your thermometer if you are an inertial observer in de Sitter. (Due to dark energy we are now at the start of a new de Sitter epoch. The present-day de Sitter temperature is the Hubble scale, $T \sim 10^{-33} eV$.)

Entropy

The area of the cosmological horizon is

$$\text{Area} = \ell^2 Vol(S_{D-2}) \tag{7.18}$$

So in $D = 4$, $S =$ area/4 gives

$$S_{D=4} = \pi\ell^2 \ . \tag{7.19}$$

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[40]See Gibbons and Hawking, "Cosmological event horizons, thermodynamics, and particle creation," 1977.

7.3 Action

The Euclidean action of dS_D is the action of S^D. This is finite and there are no boundary terms to worry about on a sphere, so it's straightforward to calculate. In $D = 4$, the Euclidean action is[41]

$$I_E = -\frac{1}{16\pi} \int_{S^D} \sqrt{g}(R - 2\Lambda) = -\frac{\ell^4}{16\pi} \left(\frac{12}{\ell^2} - \frac{6}{\ell^2} \right) Vol(S_4) = -\pi\ell^2 . \tag{7.20}$$

Curiously, this is minus the entropy, *i.e.,* the entropy is

$$S = \log Z . \tag{7.21}$$

Here's why: Recall the thermodynamic identity $\log Z = -\beta F = -\beta E + S$. The energy in GR is a pure boundary term (see next lecture!), so a compact space has $E = 0$. Thus thermodynamics predicts $\log Z = S$, and that's exactly what we found in de Sitter.[42]

de Sitter is mysterious

We will not say much more about de Sitter space in this course. A big reason for this is that we don't have any UV-complete theory of gravity in de Sitter, like we do in anti-de Sitter. We also have no clear answer to the question 'What is the de Sitter entropy?' (Does it count the microstates of something?) Since we live in de Sitter, this seems like a very important question.

Exercise: de Sitter in general dimensions
Difficulty level: straightforward, if using Mathematica to compute curvatures

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[41]Given the proliferation of things called 'S' I will start using I for action

[42] *Caveat:* In this calculation $T = 1/(2\pi\ell)$, with ℓ a fixed parameter, so the temperature is not actually a tunable variable. We can't really make sense of thermodynamics if the temperature is not a free variable. To really make sense of the discussion in this section, we need to add some additional matter to de Sitter, which allows us to tune the temperature. Then there is a true 1st law, with the temperature a tunable variable. This done in for example the nice de Sitter review hep-th/0110007. You can think of what we've described here as the limit of that calculation where the extra energy is set to zero.

Compute the entropy and the on-shell action of D-dimensional de Sitter space, and verify the relation $S = \log Z$.