6 The Gravitational Path Integral

6.1 Interpretation of the classical action

In ordinary QFT, to do a path integral we first fix the spacetime manifold M, then integrate over fields defined on M. We did the same thing in our discussion of Hawking radiation. In quantum gravity, however, we must integrate over the geometry itself. We are only allowed to specify the boundary conditions on the geometry as $r \to \infty$, just like for other fields. The gravitational path integral (in Euclidean signature) is

$$Z = \int DgD\phi\, e^{-S_E[g,\phi]}, \qquad S_E[g] = -\frac{1}{16\pi G_N}\int \sqrt{g}\,(R + \cdots) + \text{boundary terms}\ , \tag{6.1}$$

where ϕ denotes all the matter fields.

The meaning of this path integral depends on the boundary conditions, as usual. In analogy to the QFT case, we define the thermal partition function $Z(\beta)$ as the path integral on a Euclidean manifold with the boundary condition that Euclidean time is a circle of proper size β,

$$t_E \sim t_E + \beta\ , \qquad g_{tt} \to 1, \qquad \text{at infinity}\ . \tag{6.2}$$

Of course we cannot actually do the path integral. In fact, we don't even really know how to define it.[30] The best we can do is to approximate it by expanding around a classical saddlepoint, *i.e.,* a solution of the classical equations of motion:

$$Z(\beta) \approx \exp\left(-S_E[\bar{g},\bar{\phi}] + S^{(1)} + \cdots\right)\ . \tag{6.3}$$

The leading term, in which $\bar{g}, \bar{\phi}$ is a solution of the classical equations of motion, is the semiclassical approximation to the path integral. This solution must of course obey the correct boundary condition. The next term is the 1-loop term and is $O(G_N^0)$, and the dots indicate higher-loop contributions.

We already know a solution with the correct boundary conditions: the Euclidean

[30] The situation in gravity is even worse than in ordinary QFTs, since the Euclidean action is not bounded below.

Schwarzschild black hole. This is a classical saddlepoint with a Euclidean time circle of size β. Therefore, to leading approximation, the thermal free energy is the Euclidean on-shell action:

$$\log Z(\beta) \approx -S_E[\bar{g}] \ , \tag{6.4}$$

with $\bar{g}$ the Schwarzschild metric. (We have dropped $\bar{\phi}$ because no matter fields are non-zero in the Schwarzschild background.)

This partition function can be used in all of the same ways as an ordinary thermodynamic partition function. For example, recall that $\log Z = S - \beta E$, so the entropy and energy are

$$S = (1 - \beta\partial_\beta)\log Z(\beta), \qquad E = -\partial_\beta \log Z \ . \tag{6.5}$$

We will see that these agree with the area law and the black hole mass.

A similar discussion applies with an angular potential and electric potential, but we will stick to the Schwarzschild black hole to keep things simple.

Exercise: RN free energy

Difficulty: 2 lines

Using our previously calculated results for S and T (from the 1st law), and assuming energy $E = M$, find the free energy of the Reissner-Nordstrom black hole.

Exercise: RN specific heat

Difficulty: 2 lines

Compute the specific heat of Reissner-Nordstrom.

6.2 Evaluating the Euclidean action

We will now do this explicitly, in Einstein gravity (*i.e.,* no higher curvature corrections) with zero cosmological constant. It is not as simple as computing R (which vanishes for Schwarzschild!) and integrating over spacetime, since there are boundary terms to worry about and infinities to regulate.[31] Although this is entirely classical, the procedure to regulate divergences involves counterterms much like those in QFT; in fact we will see later there is a direct link between these two apparently different divergences.

Gibbons-Hawking-York boundary term

The Euclidean action is computed by first cutting off the spacetime at some large but fixed $r = r_0$. In the presence of a boundary we must add to the bulk Einstein action a boundary term, called the Gibbons-Hawking-York term (once again setting $G_N = 1$),

$$S_E[g] = -\frac{1}{16\pi}\int_M \sqrt{g}R - \frac{1}{8\pi}\int_{\partial M}\sqrt{h}K \ . \tag{6.6}$$

Here h_{ij} is the induced metric on the boundary ∂M, and the *extrinsic curvature* of ∂M is

$$K_{ij} \equiv \tfrac{1}{2}\mathcal{L}_n h_{ij} = \nabla_{(i}n_{j)} \ , \qquad K = h^{ij}K_{ij} \ , \tag{6.7}$$

with n is the inward-pointing unit normal to ∂M.[32]

The Gibbons-Hawking-York term is needed for the action to be stationary around classical solutions. The variation of the Einstein term has the schematic form

$$\delta\int_M \sqrt{g}R \sim \int_M (\text{eom})\delta g + \int_{\partial M}\left[A(g,\partial g)\delta g + B(g,\partial g)\partial\delta g\right] \ , \tag{6.9}$$

where 'eom' essentially means the Einstein tensor[33] and the boundary terms come from integrating by parts. On a classical solution, the bulk term vanishes. If we impose boundary conditions that fix the metric at $r = r_0$, then $\delta g|_{\partial M} = 0$, so the first boundary term vanishes, but the boundary term involving $\partial\delta g$ does not. The Gibbons-Hawking-York term fixes this problem. It is chosen so that the variation of

[31] This subsection follows Hawking's chapter in *General Relativity, an Einstein Centenary Survey*, Hawking and Ellis *eds*.

[32] In the simple case that the boundary is, say, at fixed r, the induced metric $h_{ij} = g_{ij}$ where i runs over the transverse directions. That is all we will need. But more generally, the projector onto ∂M is

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{6.8}$$

(as you can see by noting $n^\mu h_{\mu\nu} = 0$), and then you must define intrinsic coordinate x^i on ∂M.

[33] $(eom)\delta g \propto \sqrt{g}G^{\mu\nu}\delta g_{\mu\nu}$

the full action (6.6) has the form

$$\delta S_E[g] = \int_M (\text{eom})\delta g + \tfrac{1}{2}\int_{\partial M} \sqrt{h} T^{\mu\nu}\delta g_{\mu\nu} \ . \tag{6.10}$$

We will return to this 'stress tensor' later, but for now the important thing is just that the boundary term has been chosen to eliminate $\partial\delta g$. Thus $\delta S_E[\bar{g}] = 0$ for variations satisfying the boundary condition and $\bar{g}$ satisfying the equations of motion.

Euclidean Schwarzschild Black Hole

The Euclidean Schwarzschild solution is obtained from the ordinary Schwarzschild metric by sending $t \to -i\tau$,

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \ . \tag{6.11}$$

What was the horizon $r = 2M$ in Lorentzian signature is now just the origin of a polar coordinate system, with angular coordinate τ identified as required for regularity at the origin,

$$\tau \sim \tau + 8\pi M \ . \tag{6.12}$$

Euclidean black holes are completely smooth solutions; they do not have an interior or a singularity.

Now we want to evaluate the action. The bulk term vanishes, since the vacuum Einstein equations set $R = 0$. The boundary term, evaluated on the surface $r = r_0$, is

$$\int_{\partial M} \sqrt{h} K = \beta(8\pi r_0 - 12\pi M) \ . \tag{6.13}$$

This is infinite as we take $r_0 \to \infty$. The procedure to regulate this divergence[34] is to add a 'counterterm' to the action,

$$S_E[g] = -\frac{1}{16\pi}\int_M \sqrt{g} R - \frac{1}{8\pi}\int_{\partial M} \sqrt{h} K + \frac{1}{8\pi}\int_{\partial M} \sqrt{h} K_0 \ , \tag{6.14}$$

[34]Caveat: this version of the procedure does not always work in asymptotically flat spacetime. As far as I know there is no entirely satisfactory and unique prescription with zero cosmological constant. Things are understood better in de Sitter space, which has finite volume, or in asymptotically anti-de Sitter space, where a similar procedure always works and plays an important role in AdS/CFT.

where K_0 is the extrinsic curvature of the *same* boundary manifold ∂M, embedded in flat spacetime. This is very similar to what we do in quantum field theory, but this calculation is entirely classical. (We will see later that in anti-de Sitter space, there is a direct connection between the two ideas). Note that the counterterm depends only on data intrinsic to the boundary surface – it is not allowed to depend on $\partial_n h$.

To compute the counterterm, we embed the boundary metric

$$ds^2_{bdry} = (1 - 2M/r_0)d\tau^2 + r_0^2 d\Omega_2^2 \tag{6.15}$$

in flat space, by repeating the calculation for the flat geometry

$$ds^2_{subtraction} = (1 - 2M/r_0)d\tau^2 + dr^2 + r^2 d\Omega_2^2 \ . \tag{6.16}$$

This gives[35]

$$\int_{\partial M} \sqrt{h} K_0 = \beta(8\pi r_0 - 8\pi M + O(1/r_0)) \ . \tag{6.17}$$

This eliminates the divergence (and changes the finite term!), giving our final answer

$$S_E = \frac{\beta M}{2} = 4\pi M^2 \tag{6.18}$$

Thus the thermal partition function, or leading approximation to the path integral, is

$$Z(\beta) = \exp\left(-4\pi M^2\right) = \exp\left(-\frac{\beta^2}{16\pi}\right) \ . \tag{6.19}$$

From this we can rederive the entropy and energy using standard thermodynamics,

$$\begin{aligned} S &= (1 - \beta\partial_\beta)\log Z = 4\pi M^2 \\ E &= -\partial_\beta \log Z = M \end{aligned} \tag{6.20}$$

The entropy agrees with the area law $S = \text{Area}/4$.

[35] In this case you can get the same answer by just evaluating K with $M = 0$. However this does not always work. The correct procedure is to subtract the curvature of a boundary surface of identical intrinsic geometry, embedded in flat spacetime.

Entropy and conical defects

We have just checked this for a special case, the Schwarzschild black hole, but this always works and agrees with the area law. Roughly, the reason it is proportional to area is that we can think of the equation $(1-\beta\partial_\beta)\log Z$ as calculating the change in the classical action produced by changing the imaginary-time identification. If you smoothly deform a solution, then $\delta S_E = 0$ by the equations of motion; but if you introduce a defect, this contributes $\delta S_E = \int_{defect}(\text{something})$. Going through the details, you can derive Area/4.[36] This is also the easiest way to derive Wald's formula, which includes the corrections to the entropy from higher curvature terms in the action.

Exercise: Schwarzschild action

Difficulty: a page or two

Derive equations (6.13) and (6.18) using (6.7).

Exercise: Euclidean methods for the BTZ black hole

Difficulty: difficult, I suspect

Evaluate the on-shell action of the Euclidean BTZ black hole obtained by Wick-rotating the metric (2.23). Check that you reproduce the correct entropy and energy. (*Caveat!* What I called M in the metric is not the energy. The energy is $E = M^2/8$.)

This calculation is similar to what we just did for asymptotically-flat Schwarzschild black holes. Note that the bulk term no longer vanishes, $R - 2\Lambda \neq 0$. The full action, including the counterterm, is

$$S_E[g] = -\frac{1}{16\pi}\int_M \sqrt{g}(R-2\Lambda) - \frac{1}{8\pi}\int_{\partial M}\sqrt{h}K + \frac{a}{8\pi}\int_{\partial M}\sqrt{h}\ . \qquad (6.21)$$

Choose a to cancel the divergence; the remaining finite expression is the correct S_E.

Reference: [Balasubramanian and Kraus, hep-th/9902121].

Comment: The counterterm depends on the dimensionality of spacetime. The simple counterterm in (6.21) only works in AdS$_3$. In higher-dimensional AdS, there are more

[36] The clearest reference I know for this is section 3.1 of Lewkowycz and Maldacena, 1304.4926.]

available counterterms, for example $\int_{\partial M} R[h]$ (the intrinsic boundary curvature), and these are also required to cancel all divergences.