5 Path integral approach to Hawking radiation

5.1 Rindler Space and Reduced Density Matrices

We will use the Euclidean path integral to justify the claim in (3.23) that the Minkowski vacuum corresponds to the Rindler state $\rho_{Rindler} = e^{-2\pi H_\eta}$. Consider a 2d QFT on a line, and let the state of the full system by the Minkowski vacuum,

$$\rho = |0\rangle\langle 0| \ . \tag{5.1}$$

As argued above, this state is prepared by a path integral on a half-plane, cut on the line $t = 0$. Let us divide the line into $x > 0$ (region A) and $x < 0$ (region B). The reduced density matrix in region A is

$$\rho_A \equiv \mathrm{tr}_B\, \rho \ . \tag{5.2}$$

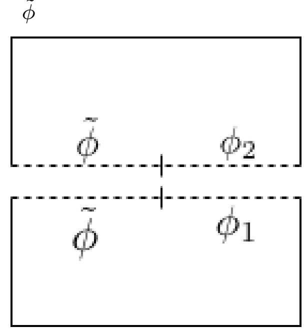
This has the nice property that all observables restricted to region A (or to the Rindler wedge that is the causal evolution of region A) can be computed from ρ_A alone:

$$\mathrm{Tr}\ \rho O(x_1)\cdots O(x_n) = \ \mathrm{Tr}\ \rho_A O(x_1)\cdots O(x_n) \ , \quad \text{for} \quad x_i > 0, |t| < x_i \ . \tag{5.3}$$

The path integral representation of ρ_A is

$$\langle\phi_2|\rho_A|\phi_1\rangle \ = \ \sum_{\tilde\phi} \langle\tilde\phi, \phi_2|0\rangle\langle 0|\phi_1, \tilde\phi\rangle \tag{5.4}$$

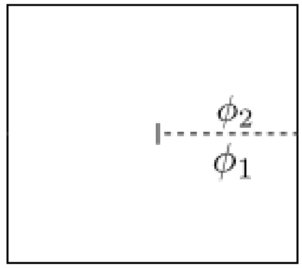
$=$



(5.5)

The upper half of this diagram corresponds to the transition amplitude $\sum_{\tilde\phi}\langle\tilde\phi, \phi_2|0\rangle$ and the lower half to the transition amplitude $\langle 0|\phi_1, \tilde\phi\rangle$. The trace sums over fields in the left Rindler wedge, which glues together these slits in the path integral, so in fact

$$\langle \phi_2 | \rho_A | \phi_1 \rangle = \tag{5.6}$$



Now comes the key observation: we can re-slice this path integral by going to polar coordinates $dR^2 + R^2 d\phi^2$, and calling ϕ 'time'. Let $H_{Rindler}$ be the operator that generates ϕ-evolution. That is,

$$\frac{1}{\hbar}[H, O] = \partial_\phi O \tag{5.7}$$

for any operator O. Then we can translate this *same* path integral back into operator language in a different way. That is, the path integral in (5.6) is equal to $\langle \phi_2 | e^{-2\pi H_{Rindler}} | \phi_1 \rangle$. Therefore

$$\rho_A = e^{-2\pi H_{Rindler}} \ . \tag{5.8}$$

This looks just like a thermal state at temperature $1/2\pi$, but it is thermal with respect to the rotation generator. When we go back to Minkowski space $\phi \to i\eta$, this becomes the boost generator corresponding to the causal development of the Rindler wedge. Therefore $H_{Rindler}$ is exactly what we called H_η above.

This is a complete path-integral derivation of the statement that the Minkowski vacuum leads to a thermal state in Rindler space. As mentioned above, this can also be shown by explicit comparison of modes, but the path integral derivation can be more useful for intuition. Another big advantage is that in the path integral derivation, we did not assume anywhere that the matter fields were free, or even necessarily weakly coupled—it is completely general.

Modular Hamiltonian

The Hamiltonian that appears in the relation $\rho_{Rindler} = e^{-2\pi H_{Rindler}}$ is a special case of a *modular Hamiltonian.* A modular Hamiltonian is simply defined as the log of a density matrix (up to normalization). It is very useful for characterizing entanglement, both in quantum gravity and in condensed matter physics.

5.2 Example: Free fields

So far, we have answered the question "What is the quantum state of fields on Rindler space?" The complete answer is equation (5.8), and does not require any mention of "particles" (which only make sense at weak coupling), or any particular observer.

However to gain a more concrete intuition for the physics it is very useful to think in terms of particles. So in this subsection we will apply to result (5.8) to free (or weakly interacting) fields, and discuss what an accelerating observer capable of detecting these particles would actually experience.

A massless free field in 2D Rindler space (in Lorentz signature) obeys the wave equation

$$\Box\Phi = \nabla_\mu \nabla^\mu \Phi = 0 \ . \tag{5.9}$$

Since η is our 'time' coordinate, we take the ansatz $\Phi = e^{-i\omega\eta} f(R)$, and find the solution

$$\Phi = e^{-i\omega\eta + ik\log R}, \qquad \omega^2 = k^2 \ , \qquad \omega > 0 \ . \tag{5.10}$$

As usual in QFT, we expand the field operator in terms of creation and annihilation operators,

$$\hat{\Phi}(\eta, R) = \int dk \left(b_k \Phi_k + b_k^\dagger \Phi_k^* \right) \tag{5.11}$$

The creation operators $b^\dagger$ create positive-energy modes ϕ_k. The b's annihilate positive-energy modes. The Rindler vacuum state is defined by

$$|0\rangle_R = b_k |0\rangle_R = 0 \ , \qquad \forall k \ . \tag{5.12}$$

It is clear that this is *not* the Minkowski vacuum state: Minkowski modes are expanded in Minkowski plane waves, and Minkowski creation and annihilation operators $a_k^\dagger, a_k$ are not the same as the Rindler ones. The fact that Rindler space has a different choice of 'time' means it has a different choice of 'energy' and therefore a different notion of 'particle' and 'vacuum':

$$\text{time coordinate} \leftrightarrow \text{energy} \leftrightarrow \text{particle} \leftrightarrow \text{vacuum}\ . \tag{5.13}$$

How does this relate to our more abstract path integral discussion above? ω in the mode-expansion (5.11) is exactly the eigenvalue of the Rindler Hamiltonian $H_{Rindler}$. That is, the 1-Rindler-particle state,

$$|k\rangle_R = b_k^\dagger |0\rangle_R \tag{5.14}$$

satisfies[19]

$$H_{Rindler}|k\rangle_R = \omega |k\rangle\ . \tag{5.15}$$

Just like in flat space there are also multiparticle states.

In the Minkowski vacuum, the quantum state of these fields is $\rho_{Rindler} = e^{-2\pi H_{Rindler}}$. We can use this to calculate observables. For example, what is the occupation number of a mode with Rindler energy $\omega = |k|$? The calculation is identical to the usual blackbody calculation:

$$\langle n_k \rangle = \frac{1}{Z}\ \mathrm{Tr}\ e^{-2\pi H_{Rindler}} b_k^\dagger b_k, \qquad Z \equiv\ \mathrm{Tr}\ e^{-2\pi H_{Rindler}} \tag{5.16}$$

The number operator $b_k^\dagger b_k$ counts the number of quanta in the mode, so it ranges from $n = 0 \dots \infty$, so

$$\begin{aligned} \langle n_k \rangle &= \left(\sum_{n\geq 0} n e^{-2\pi n|k|} \right) / \left(\sum_{n \geq 0} e^{-2\pi n |k|} \right) \\ &= \frac{1}{e^{2\pi |k|} - 1}\ . \end{aligned} \tag{5.17}$$

This is of course the Planck blackbody spectrum.

What does an observer actually *see*?

An observer who has can detect the Φ-particle will see the blackbody spectrum (5.17). However there is one last subtlety: an observer carrying a thermometer, or a calorimeter

[19] For the same reason that, in flat spacetime, if we write modes $\propto e^{-i\omega t}$ then ω is the energy.

that measures energy in, say, joules, does not measure the energy ω. In fact, ω is dimensionless, since the Rindler time coordinate is dimensionless, so this wouldn't even make sense.[20] What an observer actually calls 'energy' is the quantity conjugate to the observer's proper time. That is, the observer will consider a mode $\sim e^{-i\omega_{obs}\tau_{obs}}$ to have energy ω_{obs}, in joules or similar energy units. The proper time of a uniformly accelerating observer with acceleration a (and therefore Rindler position $R_{obs} = 1/a$) is

$$d\tau_{obs} = R_{obs} d\eta = \frac{1}{a} d\eta \ , \tag{5.18}$$

so the observer will see a mode $e^{-i\omega\eta}$ to have energy

$$\omega_{obs} = a\omega \ . \tag{5.19}$$

Accordingly, the temperature shown on an accelerating thermometer will be

$$T_{obs} = \frac{a}{2\pi} \ . \tag{5.20}$$

Aside: Transient acceleration

Strictly speaking, our discussion of accelerating observers assumes that the observer has always been accelerating, and will continue accelerating forever. Only observers who will continue accelerating forever actually have a Rindler horizon.

However, a temporarily accelerating observer will also see Unruh radiation. It does not quite make sense to talk about a 'temperature' in this case because the observer's thermometer will not reach exact equilibrium in any finite time. When the observer starts accelerating, there will be some transient effects, and then the observer will feel thermal radiation; the thermometer will start to heat up, asymptotically approaching temperature $a/2\pi$; and when the observer stops accelerating the thermometer will again experience some transient effects, then radiate and cool back down to zero temperature.

So, as long as the acceleration lasts a long time compared to the equilibration timescale $t_{equil} \sim 1/T \sim R_0$, the Unruh temperature is still meaningful in this situation. On the other hand, for short bursts of acceleration, our analysis does not apply. Instead we would need to solve a time-dependent problem. This can be done using Feynman

[20] ie $ds^2 = dR^2 - R^2 d\eta^2$, so all the dimensions are carried by R; η is like an angular coordinate.

diagrams that describe emission/absorption of particles from an arbitrary worldline. (There are many wrong papers on this topic. A correct, clear, and short paper that also has a nice derivation of the Unruh effect from Green's functions is: "Transient phenomena in the Unruh effect," Bauerle and Koning.)

5.3 Importance of entanglement

What does physics look like in the Rindler vacuum, $|0\rangle_R$? To an accelerating observer, it would look ordinary: this observer would detect no particles. A geodesic observer, however, would detect observers, since this observer must notice that fields are not in the Minkowski vacuum. As long as the geodesic observer is in the Rindler wedge, this just looks like some particular excited state. However, timelike geodesics cannot stay in the Rindler wedge forever — eventually they go through the Rindler horizon. The Rindler vacuum state is singular at the horizon. That is, the energy density measured by a geodesic observer diverges at the Rindler horizon. There is no 'beyond' the horizon in this state.

This makes sense. In the Rindler vacuum, there are *no* correlation between fields in the left and right Rindler wedges:[21]

$$_R\langle 0|\phi(x_1)\phi(x_2)|0\rangle_R = 0 \quad \text{for} \quad x_1 \in R_{left}, \quad x_2 \in R_{right} \ . \tag{5.21}$$

If there are no correlations, who's to say that these wedges are actually 'next to each other'? In a sense they are not. Thus in the vacuum state, the Rindler wedge does not extend beyond the horizon.

The key to obtaining a finite energy density on the Rindler horizon is to have a lot of entanglement between the left and right Rindler wedges. In the exercise below you will show explicitly how, in the Minkowski vacuum, the left and right Rindler wedges are maximally entangled, much like the two spins in Bell's thought experiment.[22] Any state with smooth horizon must be highly entangled across the horizon.

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[21] In this equation $|0\rangle_R$ means the product vacuum where each Rindler wedge is in its vacuum.
[22] http://en.wikipedia.org/wiki/Bell's_theorem

Exercise: Entanglement warm-up

Difficulty level: easy

Consider a quantum system consisting of two particles A and B, each with two states $|0\rangle$ and $|1\rangle$ (which you can think of as spin-up and spin-down). Suppose the full system is in the maximally entangled pure state

$$|\psi\rangle = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \ . \tag{5.22}$$

(This is sometimes called a Bell pair). Find the reduced density matrix ρ_A for particle A. You will find a mixed state. Compute the *entanglement entropy* of this mixed state, defined as

$$S_A = -\operatorname{tr}_A \rho_A \log \rho_A \ . \tag{5.23}$$

Exercise: Thermofield Double

Difficulty level: conceptual

Consider a quantum system with Hilbert space $\mathcal{H}$. Any mixed state ρ can be thought of as a pure state in an enlarged system. That is, we can always add an auxiliary Hilbert space $\tilde{\mathcal{H}}$ and find a pure state

$$|\Psi\rangle \in \tilde{\mathcal{H}} \otimes \mathcal{H} \tag{5.24}$$

such that

$$\rho = \operatorname{tr}_{\tilde{H}} |\Psi\rangle\langle\Psi| \ . \tag{5.25}$$

This is called *purifying* the mixed state. In this problem you will show that Minkowski space is a purification of Rindler space.

The Minkowski Hilbert space[23] factorizes[24] into two copies of the Rindler Hilbert space,

$$\mathcal{H}_M = \tilde{\mathcal{H}}_R \otimes \mathcal{H}_R \ , \tag{5.26}$$

which are the Hilbert spaces associated to the left Rindler $x < 0$ and right Rindler

[23] By 'Minkowski Hilbert space' we really mean Hilbert space of the theory on an infinite plane, since Hilbert spaces are defined by the *space* on which a theory lives, not the spacetime. Similarly by 'Rindler Hilbert space' we mean the Hilbert space of the theory quantized on a half-plane.

[24] This is not quite true due to UV divergences, but this doesn't matter for this problem

$x > 0$.[25] In terms of the field data, this just means that a field in Minkowski space at $t = 0$, $\phi_M(x)$, can instead be written as the pair $(\tilde{\phi}_R, \phi_R)$ where $\tilde{\phi}_R$ is a field on the left Rindler half-plane, and ϕ_R is a field on the right Rindler half-plane.

(a) The Minkowski groundstate $|0\rangle$ is formally a functional that turns field data at $t = 0$ into complex numbers. That is, the ground state wavefunction is

$$\Psi_0[\tilde{\phi}_R, \phi_R] = {}_M\langle \tilde{\phi}_R, \phi_R | 0 \rangle_M \ . \tag{5.27}$$

The subscript M means 'Minkowski', ie a state on the full space. Write down the path integral that computes this wavefunction, and draw the corresponding picture along the lines of the path integral pictures above.

(b) Now re-slice this same path integral using the Rindler Hamiltonian $H_{Rindler}$, which generates Euclidean rotations ∂_ϕ. That is, write an operator expression of the form

$$\Psi_0[\tilde{\phi}_R, \phi_R] = {}_R\langle \tilde{\phi}_R | \cdots | \phi_R \rangle_R \tag{5.28}$$

and fill in the dots with an expression involving $H_{Rindler}$.

(c) We want to show that the Minkowski state is the same as the doubled Rindler state

$$|TFD\rangle_{R\otimes R} \equiv \sum_n e^{-\beta E_n/2} |n\rangle_R |n\rangle_R^* \tag{5.29}$$

where this is a sum over Rindler energy eigenstates, E_n is the Rindler energy, $\beta = 2\pi$ is the Rindler temperature, and $*$ means CPT conjugate. This is called the thermofield double state.

To demonstrate this, check that the matrix elements of the state defined in (5.29) agree with the ones you wrote above,

$$_M\langle \tilde{\phi}_R, \phi_R | TFD \rangle_{R\otimes R} = \Psi_0[\tilde{\phi}_R, \phi_R] \ . \tag{5.30}$$

To do you this you will need to note that the mapping from Minkowski states to Rindler

[25] Clearly $\mathcal{H}_R = \tilde{\mathcal{H}}_R$ but the tilde will be useful to keep track of things.

$\otimes$ Rindler states is

$$|\tilde{\phi}_R, \phi_R\rangle_M \to |\tilde{\phi}\rangle_R^* |\phi_R\rangle_R \ , \tag{5.31}$$

where the conjugation is needed because time runs 'backward' in the left Rindler wedge.

What you have just shown is that the full Minkowski vacuum can be reinterpreted as the thermofield double in two copies of Rindler space.

(d) Finally, check that tracing over the left Rindler Hilbert B space produces a thermal state in the right Rindler Hilbert space A,

$$\rho_A \equiv \mathrm{tr}_B \, |TFD\rangle\langle TFD| = e^{-2\pi H_{Rindler}} = \sum_n e^{-2\pi E_n} |n\rangle_R \, {}_R\langle n| \tag{5.32}$$

This is an alternative derivation of the Unruh thermal state.

Reference: This problem is based on Maldacena's thermofield double interpretation of black holes in AdS/CFT [hep-th/0106112], which we will hopefully discuss later in the course.

Comment: Another way to approach this problem is to think of the infinite-temperature state $\sum_n |n\rangle_R |n\rangle_R^*$ as produced by a path integral over an infinitessimal strip in the Euclidean plain, positioned along the negative τ-axis. Then the thermofield double state is produced by evolving this by $\beta/4$ to the left and $\beta/4$ to the right, producing a state on the $\tau = 0$ line.

Exercise: State on an interval in 2d CFT

Difficulty level: a couple pages

In this exercise you will work out the state of a 2d conformal field theory in the vacuum, when restricted to a finite interval. This is similar to the Unruh state, but on a finite region instead of a semi-infinite plane (which in 2d would be a half-line).

At $t = 0$, we define region A to be the interval $x \in (0, \ell)$, and region B is everything else. Let z be a complex coordinate on 2d Euclidean space. The conformal mapping

$$z = -\frac{w}{w - \ell} \tag{5.33}$$

maps the half-line in the z-coordinate to the interval $x \in (0, \ell)$, where $w = x + it_E$. In the z coordinate, evolution of the half-line is generated by the rotational vector field ζ.

(a) Write ζ in the z coordinate,[26] then do the coordinate change (5.33) to write it in the x, t_E coordinates.

(b) The modular Hamiltonian, which generates the time evolution of the interval, is then

$$H_A = \int_A dx\ T_{t\mu}\zeta^\mu|_{t_E=0}\ , \tag{5.34}$$

where $T_{\mu\nu}$ is the usual stress tensor. Write the integrand explicitly in terms of T_{tt}, x, and ℓ.

It follows that the quantum state on the interval is

$$\rho_A = e^{-2\pi H_A}\ . \tag{5.35}$$

(c) Sketch the vector field ζ on the Euclidean (x, t_E) plane.

(d) Continue $t_E \to it$, and sketch the vector field ζ in 2d Minkowski space (x, t).

5.4 Information paradox

Reference: See Harlow's lectures on black hole information, [arXiv: 1409.1231], especially sections 4 and 5.

Suppose the universe starts in a pure state, with some very low-density matter. If this matter eventually collapses to form a black hole, then the quantum fields outside the black hole will be in the Unruh state. The black hole will radiate. This is a quantum effect, so it happens very slowly, but gradually the black hole will lose energy and therefore the mass M will decrease. After a very long time, the black hole will evaporate completely, or at least to the point where it is Planck-sized and we can no longer trust our effective field theory. See figure 2.

[26] *i.e.,* define $z = z_1 + iz_2$ and write ζ in terms of the two real coordinates z_1 and z_2.

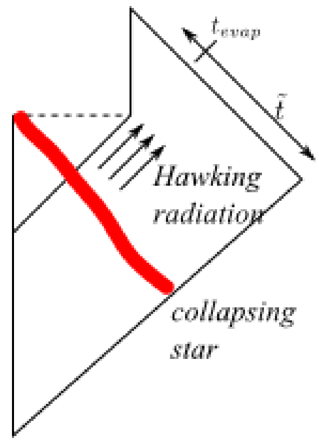


Figure 2: Penrose diagram for a black hole that forms by collapse, then evaporates.

The black hole radiation is in a thermal state, which is a mixed state. This is the *information paradox*: A pure initial state $\rho_{init} = |\psi\rangle\langle\psi|$ has evolved to a mixed state ρ_{final}. In quantum mechanics, pure states must remain pure:

$$\rho \to e^{-iHt}\rho e^{iHt} = |\psi(t)\rangle\langle\psi(t)| \ . \tag{5.36}$$

This problem is unsolved. There are roughly three options:

1. Quantum mechanics is wrong: In quantum gravity, time evolution cannot be described by $|\psi\rangle \to e^{-iHt}|\psi\rangle$. This was Hawking's conclusion. It is an entirely reasonable conclusion, forced on him by the calculation of Hawking radiation, but it means that we know next-to-nothing about quantum gravity since it is not even quantum mechanics. To argue for some other conclusion, the burden of proof is on us to show where Hawking made an error!

2. Remnants. We do not claim to know what happens in the final instants of black hole evaporation, when the black hole is Planck sized. Our effective field theory breaks down and cannot be trusted. It is possible that the black hole never evaporates completely, but leaves behind a tiny, Planck-sized object, storing all of the information that was apparently lost. Since the initial black hole could be arbitrarily large, the number of possible remnant states must be infinite. This is a serious departure from ordinary QFT, in which the spectrum of states at a given energy in a finite size box is always finite. It calls into question our ability

to make any sense of QFT – do remnants run in loops? If so, they are suppressed by M_{Planck}, but enhanced by a factor of ∞ for the infinite number of internal states.

3. Black hole evaporation is unitary, but Hawking's calculation is wrong in some subtle way. This also requires a radical departure from ordinary field theory: Hawking's calculation followed the usual rules for QFT, so if this calculation was wrong, what else are we doing wrong? Shouldn't we be able to detect this problem in other situations, too? In other words, where is the error?

Small corrections do not solve the problem

At first glance, another possibility is that Hawking's calculation, although correct to leading order, has subleading corrections that recover the information. However, this is not possible. I think this has long been understood — otherwise there would be no paradox — but the cleanest arguments, based on the entanglement between bits of Hawking radiation, are quite recent.[27]

Quantifying the information problem

We have been using vague words like 'information' so let's quantify things. Define a coordinate $\tilde{t}$ along future null infinity I^+ as in figure 2. Define $\rho_{rad}(\tilde{t})$ to be the density matrix of the quantum fields on the slice of spacetime that goes from spacelike infinity to the point $\tilde{t}$ along I^+. This is the density matrix of a subsector consisting only of the Hawking radiation that has already left the black hole at time $\tilde{t}$.

The entanglement entropy of the radiation is

$$S_{rad}(\tilde{t}) = -\operatorname{tr} \rho_{rad} \log \rho_{rad} \ . \tag{5.37}$$

This is zero in a pure state, but non-zero in a mixed state — in a thermal state, is the ordinary thermal entropy. What do we expect this to look like as a function of time? At early times, when only a few Hawking quanta have left the black hole, these quanta will look completely thermal. This is true even if the full state is pure, since we are looking at a tiny subsector. As more radiation comes out, $S_{rad}(\tilde{t})$ will increase.

[27] See Mathur [arXiv: 0909.1038] and Harlow's lectures.

Now, if the Hawking calculation is exactly right, S_{rad} just keeps increasing until the black hole evaporates completely. At the end of the day, S_{rad} is a large finite number. This is now the full system, and $S_{rad} \neq 0$, so a pure state has evolved to a mixed state.

If, on the other hand, the evaporation is unitary, then eventually S_{rad} must start to decrease. This happens when a large fraction of the radiation has escaped, and, given sufficient precision, we would in principle be able to detect tiny correlations in the outgoing Hawking radiation. In this scenario, by the time we reach $\tilde{t}_{evap}$, the entanglement entropy must go back down all the way to zero so that we end in a pure state.

See Harlow's lectures (section 5) for more details and a plot of the hypothesized entanglement entropy as a function of time in this scenario.

AdS/CFT

There is no satisfactory resolution of the information paradox. A definitive resolution requires a UV-complete theory of quantum gravity. One such theory comes from AdS/CFT, as we'll discuss later. In this case, the duality strongly suggests that black hole evaporation is unitary. However, it does not resolve the paradox, because we still do not know what went wrong with Hawking's calculation. AdS/CFT only addresses the problem indirectly, by mapping it do a different more tractable problem, without telling us what broke down in our effective field theory. It is likely that there are subtle violations of locality in quantum gravity, and that this is responsible. However this remains poorly understood.

5.5 Hartle-Hawking state

We have seen there is no unique vacuum state in quantum field theory. The same is true on a black hole background. A natural state to consider, which is analogous to the vacuum state we defined in Minkowski space, is a state prepared by a path integral on the analytically continued Euclidean spacetime,

$$ds^2 = (1 - 2M/r)d\tau^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega_2^2 \ . \tag{5.38}$$

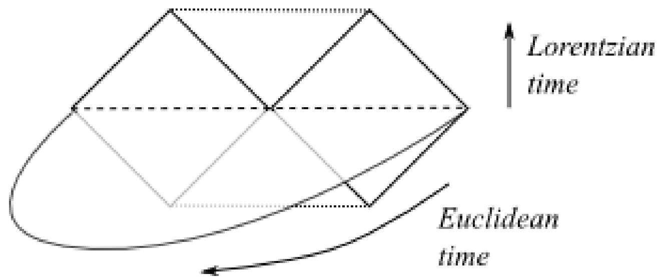


Figure 3: Schwarzschild spacetime. The Euclidean path integral produces a pure, highly entangled state on the two-sided Lorentzian spacetime. The quantum state on the right half of the Penrose diagram, where we live, is therefore mixed. This reduced state is the Hartle-Hawking thermal state.

with the imaginary-time identification $\tau \sim \tau + \beta$. This spacetime only has $r > 0$, there is no interior. The $t = 0$ slice of the Lorentzian spacetime is the $\tau = 0$ slice of the Euclidean spacetime, see figure 3.

Sending $\tau \to \tau + \beta/2$ takes us to the other side of the Penrose diagram in the maximal analytic extension of Schwarzschild. This can be shown in detail using Kruskal coordinates. A simpler way to see this is to go to Rindler coordinates near the horizon. By changing these Rindler coordinates to Minkowski-like coordinates good near the horizon, we can continue through the horizon to the other side of the Penrose diagram. So, just like in Rindler space, we get to the other side of the horizon by going half way around the Euclidean circle.

This path integral prepares an entangled state on $\tilde{M} \times M$, the product of the left and right Minkowski spaces. Just as in Rindler space, the reduced density matrix on our spacetime M will be a mixed state,

$$\rho_{HH} = e^{-\beta H} \tag{5.39}$$

where H is the ordinary Minkowski Hamiltonian associated to time translations ∂_t. This is called the 'Hartle-Hawking state.' It describes a black hole in equilibrium with a bath of radiation outside the black hole.

This is not the only state we could consider. See note 3 for a discussion of other possibilities. Hawking showed that a black hole formed by collapse will end up in the 'Unruh state', which is a state where the black hole radiates into a cold outside region.

Greybody factors

The Hartle-Hawking vacuum (5.39) is time-independent. This means that, in each mode, the flux of outgoing Hawking radiation is equal to the flux of ingoing radiation.

A mode ϕ_k outside the black hole does not necessarily fall in; it is absorbed with probability given by the absorption cross-section $\sigma_{abs}(k)$. Therefore, the only way a black hole can be in thermal equilibrium with a bath at temperature T is if the Hawking emission measured at infinite is actually

$$\langle n_k \rangle = \frac{1}{e^{\beta w} - 1} \sigma_{abs}(k) \ . \tag{5.40}$$

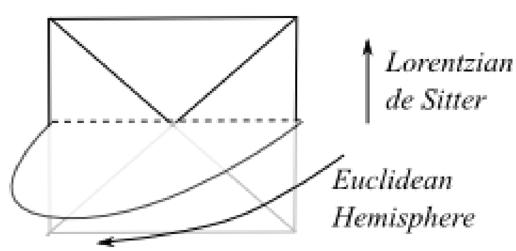
The extra factor is the 'greybody factor'. We will probably calculate some greybody factors later.

Aside: Cosmology

If the early universe is described by inflation, then it is the story of a slowly evolving de Sitter spacetime. De Sitter spacetime is the Lorentzian continuation of a sphere. That is, the metric of Euclidean de Sitter is just

$$ds^2 = d\Omega_D^2 \ . \tag{5.41}$$

The equator of the S^D is the $t = 0$ slice of global de Sitter space:



The state of quantum fields during inflation is responsible for present-day observables including the primordial temperature fluctuations in the CMB, observed by experiments like COBE, WMAP, and Planck. Since there is no unique vacuum, we must pick a state of the quantum fields in de Sitter. For various reasons,[28] we usually

[28]Here are some reasons: (1) This state respects the symmetries of de Sitter; (2) At short distances, this vacuum is the one in which comoving observers see no particles (ie it coincides locally with the

assume this state is the so-called 'Euclidean vacuum', also called the 'Bunch-Davies vacuum' or various other things. This state is prepared by a Euclidean path integral on the hemisphere, cut along the equator. Therefore this quantum state, unlike the Hartle-Hawking state, has quite possibly already been observed experimentally.

Exercise: Decay of Schwarzschild

Difficulty: a couple pages

(a) A black hole in asymptotically flat spacetime loses energy via Hawking radiation. If the initial mass is M, how long before the black hole radiates away completely?

(b) How heavy, in solar masses, would a black hole need to be for its lifetime to be the age of the universe $t \sim 13$ billion years?

(If such black holes exist, we might be able to observe the final moments of decay, when a large burst of energy is released in Hawking radiation. Unfortunately there is no particularly good reason to think they should exist, since black holes formed by stellar collapse must have $M_{initial} \gtrsim$ a fewM_{sun}.)

(c) What is the typical energy (in eV) of a particle emitted from a solar mass black hole via Hawking radiation?

Exercise: Superradiance

Difficulty: a few lines

Rotating (Kerr) black holes are labeled by mass M and angular momentum J, or equivalently by a temperature T and angular potential Ω. The spacetime is rotationally invariant and stationary, so modes of a scalar field can be written $\phi \sim e^{-i\omega t + im\phi} S_n(r,\theta)$, where n labels the solutions of given ω, m.[29] The Hawking decay rate of a rotating black hole is

$$\Gamma_{\omega,m,n} = \frac{1}{e^{\beta(\omega - m\Omega)} - 1}\sigma_{abs}(\omega, m, n) \tag{5.42}$$

Minkowski vacuum); (3) at late times, due to the cosmological expansion, any state will dilute into this state.

[29] Actually, the wave equation fully separates, so in fact $S(r,\theta) = R(r)F(\theta)$. This is surprising and nontrivial, since the background has only two Killing vectors. Similarly, the geodesic equation on Kerr has an 'extra' conserved quantity.

(a) Take the zero-temperature limit of (5.42). (*Hint:* $\omega > 0$ and m is any integer. The answer should not be trivial.)

(b) For this decay rate to make any sense, what can you conclude about σ_{abs}?

Your conclusion is a phenomenon called 'superradiance.' It is a wave analogue of the Penrose process discussed previously. In this exercise we took the Hawking formula as our starting point, but the result is entirely classical – you would reach the same conclusion by solving the wave equation on the Kerr background, and treating the black hole scattering experiment as a 1d quantum mechanics barrier transmission problem.

Superradiance very efficiently converts rest mass to radiation energy. It is believed to be responsible for the absurdly high luminosity of *quasars*: a single quasar consisting of a highly rotating black hole surrounded by infalling matter has roughly the luminosity of the entire Milky Way (10^{11} stars!).[30]

[30] More accurately, a close cousin of superradiance involving magnetic fields. The details of how this works are still largely unknown.