4 Path integrals, states, and operators in QFT

To put our derivation of Hawking radiation on a solid footing, and for other applications to gravity later on, we will now take a slight detour to explain the relationship between path integrals and states in quantum field theory. (This is material not normally covered in detail in QFT courses or books; it is assumed that the reader is already familiar with path integrals at the level of Peskin and Schröder.)

4.1 Transition amplitudes

Path integrals define transition amplitudes. A Euclidean path integral defines a transition amplitude under evolution by $e^{-\beta H}$:

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_2} D\phi \ e^{-S_E[\phi]} \ . \tag{4.1}$$

This involves a split into space and (Euclidean) time; $\phi_{1,2}$ is a boundary condition that specifies data at a fixed time. Exactly what this path integral means depends on the topology of space. If space is a plane (or line in 2d), then we depict this by

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = \tag{4.2}$$

meaning it's a Euclidean path integral over an infinite strip $R^{d-1}\times$interval, with the boundary conditions shown and the interval has length β.

If space is a sphere (or circle in 2d), then the appropriate path integral is

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (4.3)$$
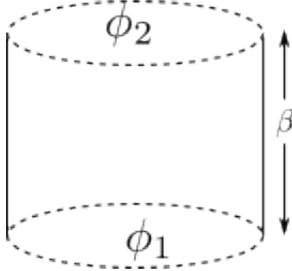
ie it is a path integral over a cylinder $S^{d-1} \times$interval, of length β.

4.2   Wavefunctions

The transition amplitude defines the wavefunction, in the Schroedinger picture. For example the wavefunction for the state

$$|\Psi\rangle = |\phi_1(\tau)\rangle = e^{-\tau H} |\phi_1\rangle \qquad \qquad (4.4)$$

is the overlap

$$\Psi[\phi_2] \equiv \langle \phi_2 | \Psi \rangle . \qquad \qquad (4.5)$$

4.3   Cutting the path integral

A 'cut' is a spatial slice of the Euclidean manifold. It is a codimension-1 surface Σ. To define the transition amplitude, we specified data on two cuts, at $\tau = 0$ and $\tau = \beta$. We can formally think of a path integral with one set of boundary conditions and one open cut as a quantum state. That is, the state

$$|\Psi\rangle = e^{-\beta H} |\phi_1\rangle \qquad \qquad (4.6)$$

is the path integral

$$|\Psi\rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=??} D\phi \; e^{S[\phi]} = \qquad\qquad\qquad \text{(4.7)}$$

This is a formal object where the data on the top cut is left unspecified. It is a functional $|\Psi\rangle$ that turns field data $\langle\phi_2|$ into complex numbers $\langle\phi_2|\Psi\rangle$.

More generally, any path integral with an open cut Σ defines a quantum state on Σ. For example, this Euclidean path integral in a 2D QFT defines some particular state on a circle, $\Sigma = S^1$:

$$|X\rangle = \qquad\qquad\qquad\qquad \text{(4.8)}$$

The wavefunction of this state is computed by the path integral

$$X[\phi_2] = \langle\phi_2|X\rangle = \qquad\qquad\qquad\qquad \text{(4.9)}$$

We could also insert some operators into this path integral to get a different state:

$$|X'\rangle = \qquad\qquad\qquad\qquad\qquad\qquad (4.10)$$

This means a Euclidedan path integral weighted by $O_1(x_1)O_2(x_2)e^{-S_E[\phi]}$, instead of just the usual $e^{-S_E[\phi]}$.

4.4   Euclidean vs. Lorentzian

So far we have discussed Euclidean path integrals. But states are states: they are defined on a spatial surface and do not care about Lorentzian vs Euclidean. The state $|X\rangle$, defined above by a Euclidean path integral, is a state in the Hilbert space of the Lorentzian theory. It is defined at a particular Lorentzian time, call it $t = 0$. It can be evolved forward in Lorentzian time by acting with e^{-iHt}, or equivalently by performing the Lorentzian path integral:

$$|X(t)\rangle = e^{-iHt}|X\rangle = \qquad\qquad\qquad\qquad (4.11)$$

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Since $|X\rangle \equiv |X(0)\rangle$ was defined by a Euclidean path integral, the state $|X(t)\rangle$ is a path integral that is part Euclidean, part Lorentzian:

$$|X(t)\rangle = \qquad\qquad\qquad\qquad\qquad\qquad (4.12)$$

Again, this equation should be read as a formal definition of the state that tells you what path integral to perform to compute transition amplitudes:

$$\langle \phi_2 | X(t)\rangle = \qquad\qquad\qquad\qquad\qquad (4.13)$$

4.5   The ground state

Evolution in Euclidean time damps excitations. Suppose we start in some state $|Y\rangle$ and expand in energy eigenstates:

$$|Y\rangle = \sum_n y_n |n\rangle, \qquad H|n\rangle = E_n |n\rangle . \qquad\qquad (4.14)$$

Then by evolving over a long Euclidean time we can project onto the lowest energy state,

$$e^{-\tau H}|Y\rangle \approx e^{-\tau E_0} y_0 |0\rangle . \qquad (\tau \to \infty) \qquad\qquad (4.15)$$

It follows that we can define the (unnormalized) ground state by doing a path integral that extends all the way to infinity in one direction. For example the ground state on

the line is produced by the Euclidean path integral

$$|0\rangle_{line} = \qquad\qquad\qquad\qquad\qquad (4.16)$$

$$\infty$$

This means a path integral on the semi-infinite plane, with an open cut at the edge. The ground state on a circle is produced by the path integral on a semi-infinite Euclidean cylinder,

$$|0\rangle_{circle} = \qquad\qquad\qquad\qquad\qquad (4.17)$$

$$\infty$$

These states are unnormalized.

4.6   Vacuum correlation functions

Path integrals with cuts can be glued together to make transition amplitudes. For example, for a theory on a line, the vacuum-to-vacuum amplitude is

$$\infty$$

$$\langle 0|0\rangle = \int D\phi\, e^{-S_E[\phi]} = \qquad\qquad\qquad\qquad (4.18)$$

$$-\infty$$

The lower half-plane produces $|0\rangle$, the upper half-plane produces $\langle 0|$, and glueing them together along the cuts at $\tau = 0$ produces the transition amplitude. One way to see that we should glue is to insert the identity:

$$\langle 0|0\rangle = \sum_{\phi_1} \langle 0|\phi_1\rangle\langle \phi_1|0\rangle \ . \qquad\qquad\qquad (4.19)$$

The first term is a path integral on the upper half plane; the second term is a path integral on the lower half plane; and summing over all possible boundary conditions ϕ_1 in the middle just says that fields should be continuous across $\tau = 0$ and therefore glues the half-planes together.

Expectation values of local operators are computed by similar path integrals, but with extra operator insertions. For example, correlation functions are expectation values in the vacuum state. In Euclidean signature these are computed by the path integral

$$\langle O_1(x_1)O_2(x_2)\rangle \equiv \langle 0|O_2(x_2)O_1(x_1)|0\rangle \tag{4.20}$$

$$= \tag{4.21}$$

This picture means the path integral of $O_1(x_1)O_2(x_2)e^{-S_E[\phi]}$ over fields on R^d. (The ordering of operators does not matter on the lhs of (4.20), but is important on the rhs; more in this below.)

Time-ordered Lorentzian vacuum correlation functions are computed by a more complicated, 'folded' path integral that is part Euclidean and part Lorentzian. For example (assuming $t_1 > t_2$),

$$\langle O_1(t_1, \vec{x}_1)O_2(t_2, \vec{x}_2)\cdots\rangle = \langle 0|\left(e^{iHt_1}O_1(0, \vec{x}_1)e^{-iHt_1}\right)\left(e^{iHt_2}O_2(0, \vec{x}_2)e^{-iHt_2}\right)|0\rangle \tag{4.22}$$

is computed by the following path integral:

$$(4.23)$$
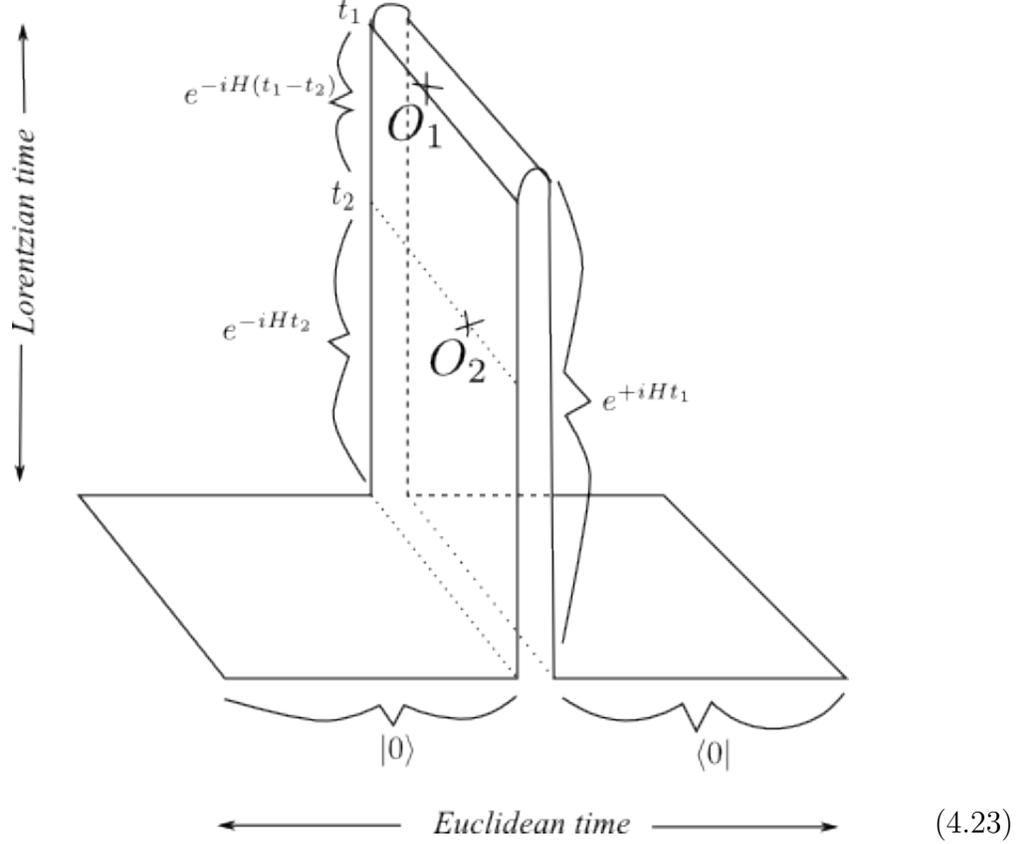
This path integral starts at $t = -i\infty$ on the left; evolves to $t = 0$ to prepare the vacuum state; evolves in Lorentzian time to $t = t_2$, where O_2 is inserted; then evolves to t_1 where O_1 is inserted; then evolves backwards in Lorentzian time to $t = 0$; then evolves to $t = +i\infty$ for the vacuum 'bra'. Again, this picture means you should do the path integral

$$\int D\phi \, O_1(t_1, \vec{x}_1) O_2(t_2, \vec{x}_2) e^{iS[\phi]} \qquad (4.24)$$

where we integrate over all fields ϕ defined on the mixed-signature manifold in the picture. The Lorentzian action appears in this expression; when you integrate over the Euclidean part of the manifold, the fact that t is imaginary will automatically change this into $e^{-S_E[\phi]}$.

We rarely need to think about doing folded path integrals like (4.23). Instead, we do one of two equivalent things: (1) We compute the Euclidean path integral with arbitrary values of the insertion points, then analytically continue to Lorentzian time,

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or (2) We use an $i\epsilon$ prescription to compute the Lorentzian path integral. Actually the usual $i\epsilon$ prescription is just a deformation of the integration contour (that is, integration contour in field space) shown in figure (4.23), and computes exactly the same quantity. So if you're ever wondered what you were doing with that $i\epsilon$, the answer is the figure in (4.23)!

Aside: Some comments on time ordering

In a sense, time ordering does not really exist in Euclidean signature: fields commute,

$$\langle \cdots O_1(x_1) O_2(x_2) \cdots \rangle = \langle \cdots O_2(x_2) O_1(x_1) \cdots \rangle . \tag{4.25}$$

One way to see this is to note that correlators computed by the path integral

$$\int D\phi \, O_1(x_1) O_2(x_2) \cdots e^{-S_E} \tag{4.26}$$

are just statistical averages, so they commute just like observables in stat-mech. Put differently, the *reason* that fields don't commute in Lorentzian signature is because the correlator is not an analytic function of the coordinates. It has branch cuts when O_2 hits the light-cone of O_1, and requires an $i\epsilon$ prescription to define the function. Different choices of $i\epsilon$ prescription give different types of correlation functions, and we denote these different choices by writing the fields in a different order. In Euclidean signature, correlators are analytic, there are not branch cuts, and there are no $i\epsilon$'s, so we don't have to worry about how fields are ordered.

However, when we cut the path integral to translate to operator language, the field operators don't commute, even in Euclidean signature. They are 'time'-ordered according to whatever slicing we choose for the path integral. So if states are defined on constant-Euclidean-time slices, the path integral translates into an operator expression with fields ordered according to their Euclidean time. If states are defined on constant-r slices (as we often do in conformal field theory), then the corresponding operator expression has radially-ordered fields.

4.7 Density matrices

A density matrix is an operator; it takes a bra and a ket, and produces a complex number. Thus any path integral with two open cuts defines a density matrix (unnormalized). For example, the density matrix $\rho = e^{-\beta H}$, for a theory on a circle, is formally the doubly-cut Euclidean path integral

$$\rho \equiv e^{-\beta H} = \quad\quad\quad \beta \quad . \tag{4.27}$$
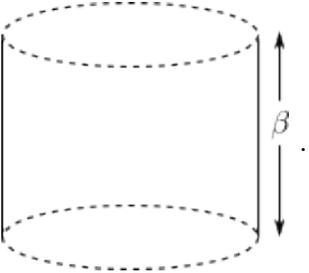
This is just a picture representing the statement that matrix elements $\langle \phi_2 | \rho | \phi_1 \rangle$ are computed by the path integral with boundary conditions $\phi_{1,2}$ on the cuts.
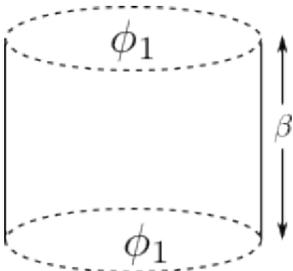

4.8 Thermal partition function

The density matrix $\rho = e^{-\beta H}$ is the density matrix in a thermal ensemble at temperature $T = 1/\beta$. The thermal partition function is
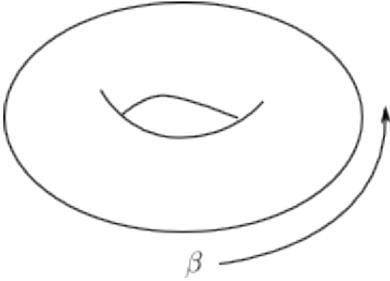
$$Z(\beta) = \text{tr } e^{-\beta H} \ . \tag{4.28}$$

This can be represented by a Euclidean path integral as follows:

$$Z(\beta) \quad = \quad \text{tr } e^{-\beta H} \tag{4.29}$$

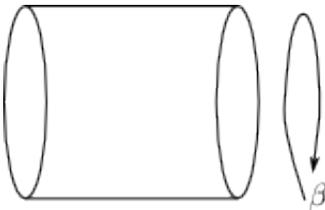$$= \quad \sum_{\phi_1} \langle \phi_1 | e^{-\beta H} | \phi_1 \rangle \tag{4.30}$$

$$= \quad \sum_{\phi_1} \quad \phi_1 \quad \beta \quad \phi_1 \tag{4.31}$$

In the last line, by summing over ϕ_1 we are really just imposing periodic boundary conditions on the cylinder. This glues together the two ends of the cylinder, producing a torus. So the thermal partition function for a 2D theory on a circle is equal to a path integral on a torus:

$$\text{on circle:} \ \ Z(\beta) = \qquad\qquad\qquad\qquad\qquad \text{(4.32)}$$

Similarly, the thermal partition function for a 2D theory on a line is computed by a path integral on an infinitely long cylinder of period β:

$$\text{on line:} \ \ Z(\beta) = \qquad\qquad\qquad\qquad\qquad \text{(4.33)}$$

The trace 'glues together' parts of the Euclidean manifold that computes ρ.

The same thing works in higher-dimensional QFT at finite temperature: If space is a plane R^{d-1}, the thermal partition function is the path integral on $R^{d-1} \times S^1$, and for a theory on S^{d-1}, the thermal partition function is a path integral on $S^{d-1} \times S^1$.

4.9 Thermal correlators

Equal-time correlators at finite temperature are defined (up to normalization) by

$$\langle O_1(t=0,\vec{x}_1)O_2(t=0,\vec{x}_2)\cdots\rangle_\beta \equiv \ \text{Tr} \ e^{-\beta H}O_1(0,\vec{x}_1)O_2(0,\vec{x}_2)\cdots \qquad \text{(4.34)}$$

By the same logic, this is computed by a path integral on a cylinder $R^{d-1} \times S^1$ (if space is a plane) or on $S^{d-1} \times S^1$ (if space is a sphere).

To compute different-time Lorentzian correlators at finite temperature, the easist method is usually to compute the Euclidean correlators first, as functions of arbitrary insertion points on the Euclidean cylinder, then analyatically continue.

Finite-temperature correlators in 2d CFT

Difficulty level: moderate, a couple pages

In a 2d conformal field theory, the 2pt function on the Euclidean cylinder of size β is fixed entirely by conformal invariance. Let $w = x + it_E$ be a complex coordinate on the Euclidean cylinder (t_E is Euclidean time and x is space). Then the 2pt function on the cylinder is

$$\langle O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) \rangle_\beta = \left(\frac{1}{\sinh(2\pi(w_1 - w_2)/\beta) \sinh(2\pi(\bar{w}_1 - \bar{w}_2)/\beta)} \right)^\Delta \quad (4.35)$$

where Δ is called the scaling dimension of the operator O.

(a) Draw a picture of the path integral on the cylinder that computes (4.35).

(b) Translate your picture into operator language. Compare to (3.10). (Don't worry about the overall sign, this is a convention.)

(c) Check that the 2pt function written in (4.35) indeed has the periodicity of a thermal correlator (see discussion around (3.10)).

(d) Analytically continue to find the finite-temperature 2pt function at real (Lorentzian) times $\langle O(t_1, x_1) O(t_2, x_2) \rangle$, where t is Lorentzian time. Don't worry about which Lorentzian ordering you are computing, just pick one. (The most obvious continuation will compute the time-ordered Lorentzian correlator.)

(e) Fix $t_1 = x_1 = 0$. Draw a picture of the complex-t_2 plane showing the singularities of (4.35). When you analytically continued in part (d), you implicitly chose a contour in this plane to define the analytic continuation. Check that if (t_2, x_2) lies inside the future light-cone of (t_1, x_1), then the analytic continuation is ambiguous, due to one of the poles in the complex-t_2 plane. This ambiguity is why timelike separated fields in Lorentzian signature do not commute.