22 Holographic entanglement at finite temperature

In this section we will discuss the HEE computation in a black hole spacetime. For explicitness, we will talk about the BTZ black hole in AdS_3, but all of the results hold qualitatively in higher dimensions, too.

The BTZ metric is static, so we need only the fixed-time metric ($\ell = G = 1$)

$$ds^2 = \frac{dr^2}{r^2 - 8M} + r^2 d\phi^2 \ . \tag{22.1}$$

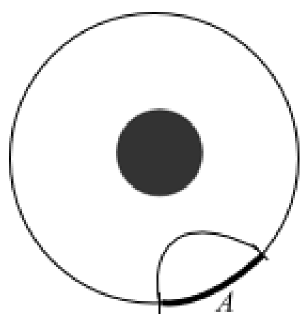
This is dual to a finite-temperature state $\rho = e^{-\beta H}$ with temperature

$$T = \sqrt{8M}/2\pi \ . \tag{22.2}$$

Choose region A to be the boundary segment

$$A : \qquad \phi \in (0, R) \ . \tag{22.3}$$

So the figure is



(22.4)

There are many geodesics connecting the endpoints of region A. In fact there are an infinite number, labeled by the integer number of times that the geodesic winds the black hole. If $R \ll 2\pi$, then there is one that is obviously the shortest, which does not wind the black hole. This is the one drawn in the figure. The length of this geodesic is infinite, but if we impose a cutoff at $r = 1/\epsilon_{UV}$, the resulting entropy is

$$\begin{aligned} S_A^{(0)} &= \frac{\text{length}(\gamma_A^{(0)})}{4G_N} \\ &= \frac{c}{3} \log\left[\frac{1}{\pi T \epsilon_{UV}} \sinh(\pi R T)\right] \ , \end{aligned} \tag{22.5}$$

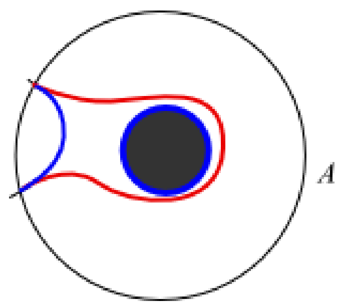
with $c = 3\ell/2G_N$.

For $0 < R \ll 2\pi$, this is the final answer. For $R > 2\pi$, the same formula (22.5) computes the length of a geodesic that winds (possibly multiple times) around the horizon. The winding geodesics do not satisfy the homology condition, *i.e.,* they cannot be continuously deformed to A. But we must also consider disjoint geodesics. For example, the horizon itself is a geodesic $\mathcal{H}$. This can be added to to the wrapped geodesic $\gamma^{(1)}$. The union

$$\gamma^{(1)} \cup \mathcal{H} \tag{22.6}$$

is homologous to region A, if we view the orientation of $\mathcal{H}$ as opposite that of $\gamma_A^{(1)}$.

If region A is large, we must choose the dominant (minimal area) surface. We are choosing between the red (wrapping) and blue (disjoint) surfaces in this figure:



(22.7)

The length of the red curve is what we computed above,

$$S_A^{red} = \frac{c}{3} \log \left[\frac{1}{\pi T \epsilon_{UV}} \sinh(\pi R T) \right] \ . \tag{22.8}$$

The length of the blue curve gets a contribution from the short part, and a contribution from the horizon:

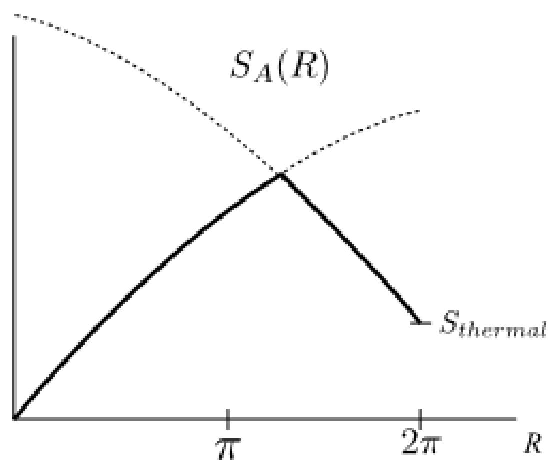
$$S_A^{blue} = \frac{c}{3} \log \left[\frac{1}{\pi T \epsilon_{UV}} \sinh(\pi(2\pi - R)T) \right] + \frac{c}{3} 2\pi^2 T \tag{22.9}$$

The first term is the answer above, applied to A^C. The second term is the thermal entropy, which we know is area(horizon)/4.

The final answer is

$$S_A(R) = \min \left[S_A^{red}, \quad S_A^{blue} \right] \tag{22.10}$$

The two contributions exchange dominance at some point $R_* > 2\pi$. At this point there is a sharp transition in the behavior of the entanglement entropy. The plot as a function of system size is something like the solid line in this figure:



(22.11)

Pure vs mixed

Clearly with a black hole, $S_A \neq S_{A^C}$ due to the homology condition. In fact we found (for $R > R_*$)

$$S_A = S_{A^C} + S_{thermal} \ . \tag{22.12}$$

Since $S_{thermal}$ is the von Neumann entropy of the full space, this can be written

$$S_A = S_B + S_{AB} \tag{22.13}$$

Thus the entanglement entropy of the black hole saturates the Araki-Lieb triangle inequality,

$$S_{AB} \geq |S_A - S_B| \ . \tag{22.14}$$

This is a special feature of thermal states in holographic systems.

22.1 Planar limit

The infinite-volume limit of the CFT is a limit of the black hole where the horizon becomes planar. We can take this limit for BTZ by assuming

$$TR \gg 1, \qquad R \ll 2\pi \ . \tag{22.15}$$

In this limit, our answer (22.5) reduces to

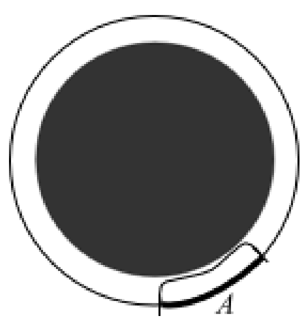
$$S_A \approx \frac{c}{3}\pi TR + \text{subleading} \tag{22.16}$$

This is equal to the thermal entropy density,

$$S_A \approx S_{thermal}(\beta) \times \frac{R}{2\pi} \ . \tag{22.17}$$

This makes sense: in the thermodynamic limit, the state is very mixed up, and the subsystem itself just looks thermal at temperature β.

Geometrically, the reason behind (22.17) is that, in this limit, the extremal surface 'hugs' the black hole horizon for most of its length:



(22.18)

The contribution from the horizon is proportional to the horizon area, *i.e.,* to the thermal entropy.

We have discussed BTZ, but the same feature generalizes to planar horizons in higher dimensions: for $L_A \gg \beta$, the extremal surface hugs the black hole horizon, giving a volume-law contribution to the finite-temperature entanglement entropy.