21 Holographic Entanglement Entropy

21.1 The formula

We now turn to entanglement entropy in CFTs with a semiclassical holographic dual. That is, we assume the CFT has a large number of degrees of freedom $N_{dof} \gg 1$ (so that $\ell_{AdS} \gg \ell_{Planck}$) and a sparse low-lying spectrum (to suppress higher curvature corrections, *i.e.*, $\ell_{AdS} \gg \ell_{string}$). We also assume that the CFT is in a state ρ with a geometric dual. This last assumption is needed since even in a holographic CFT, not every state corresponds to a particular geometry (consider, for example, a linear superposition of two black hole microstates with very different energies).

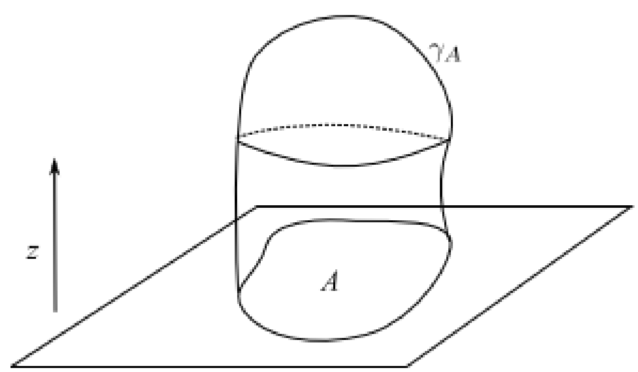
The entanglement entropy in this case is given by the *holographic entanglement entropy formula*:

$$S_A = \frac{\text{area}(\gamma_A)}{4G_N} \ , \tag{21.1}$$

where γ_A is a codimension-2, spacelike extremal surface in the dual geometry, anchored to the AdS boundary such that

$$\partial\gamma_A = \partial A \ . \tag{21.2}$$

An extremal surface is a surface of extremal area. This looks roughly as follows, with z the radial direction in AdS:



(21.3)

γ_A lives in a particular spacelike slice, so that is what is drawn here with the orthogonal (time) direction suppressed.

Two additional comments: First, in (21.1), we are only allowed to include extremal surfaces γ_A which are homologous (continuously deformable) to region A. Second, if there are multiple extremal surfaces satisfying the homology condition, then the rule is to apply (21.1) to the one which has minimal area.

History and Nomenclature

In the static case, this is generally referred to as the Ryu-Takayanagi formula, after the authors who conjectured it in 2006. In time-dependent geometries it's called the HRT formula, after Hubeny, Rangamani, and Takayanagi. Important refinements, discussed below, were also made by Headrick, and many others. The static formula, and the time-dependent formula in certain special states, were derived from the AdS/CFT dictionary $Z_{cft}(M) = Z_{grav}(\text{bdry} = M)$ by Lewkowycz and Maldacena in 2013. Because RTHHRTLM is a mouthful, I will refer to the general formula (21.1) as the HEE (holographic entanglement entropy) formula.

Static case

In a static geometry there is a natural t coordinate, and symmetry implies that γ_A will always lie within a fixed-t slice. An extremal surface in a fixed-t slice is the same as a 'minimal area surface' inside this slice, so in this case the HEE formula reduces to finding a minimal-area surface in a $d-1$-dimensional space geometry.

Extremal surfaces are minimal-area with respect to deformations inside a fixed-t slice, but maximal-area with respect to deformations in the t direction (since we can always reduce the area of a surface by making it 'more null'). The same is true of spacelike geodesics, which extremize the length of a curve in spacetime, rather than minimizing or maximizing it.

21.2 Example: Vacuum state in 1+1d CFT

Consider a 2d CFT in vacuum. Let region A by an interval of length L_A,

$$x \in [-\frac{L_A}{2}, \frac{L_A}{2}] \ . \tag{21.4}$$

The dual geometry is empty AdS_3, with metric

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2 + dx^2 + dz^2) \ . \tag{21.5}$$

z is the radial direction, with the boundary at $z = 0$.

The state is static, so we can set $t = 0$. A codimension-2 extremal 'surface' in AdS_3 is one-dimensional, *i.e.,* a geodesic. So the HEE formula instructs us to find a spacelike geodesic, in the space geometry

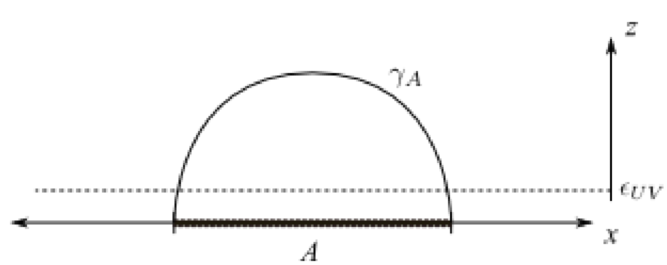
$$ds^2 = \frac{\ell^2}{z^2}(dx^2 + dz^2) \ , \tag{21.6}$$

connecting the points

$$P_1 = (z_1, x_1) = (0, -\frac{L_A}{2}) \quad \text{and} \quad P_2 = (z_2, x_2) = (0, \frac{L_A}{2}) \ . \tag{21.7}$$

However, a geodesic that reaches the boundary like this will have infinite length, since $\int \frac{dz}{z} = \infty$. This is the gravity dual of the statement that entanglement entropy in QFT is UV divergent. To regulate the divergence, we follow the same procedure we used to regulate the on-shell action, or holographic correlation functions: cut off the spacetime at $z = \epsilon_{UV}$.

Thus we want to compute the length of this curve:



(21.8)

Parameterizing the curve $(x(\lambda), z(\lambda))$ by z, the regulated geodesic length is

$$\begin{aligned} Length &= \int ds \\ &= 2L_A \int_{\epsilon}^{z_{max}} \frac{dz}{z} \sqrt{x'(z)^2 + 1} \end{aligned} \tag{21.9}$$

The factor of 2 is because we the geodesic goes out, and comes back, and we will only integrate $z \in [\epsilon, z_{max}]$ once. Treating (21.9) as a 1d "action", it is easy to show that the geodesic is a semicircle,

$$x = \frac{L_A}{2}\cos\lambda, \quad z = \frac{L_A}{2}\sin\lambda, \quad \lambda \in (\frac{\epsilon}{L_A}, \pi - \frac{\epsilon}{L_A}) \ . \tag{21.10}$$

Plugging this back into (21.9) and doing the integral gives

$$Length = 2L_A \log\left(\frac{L_A}{\epsilon_{UV}}\right) \ . \tag{21.11}$$

Therefore, applying the HEE formula (21.1),

$$S_A = \frac{L_A}{2G_N}\log\left(\frac{L_A}{\epsilon_{UV}}\right) \ . \tag{21.12}$$

The map between gravity parameters and CFT parameters in AdS_3/CFT_2 is

$$c = \frac{3\ell}{2G_N} \ , \tag{21.13}$$

where c is the central charge, so

$$S_A = \frac{c}{3}\log\left(\frac{L_A}{\epsilon_{UV}}\right) \ . \tag{21.14}$$

This agrees perfectly with our general discussion of the structure of entanglement entropy in QFT in even spacetime dimensions, (19.6), including the UV divergence. The prefactor also agrees exactly with the known result in 2d CFT, (19.8).

Exercise: 2d HEE

Fill in all the missing steps — *i.e.,* solve for the geodesic and do the length integral — in the derivation of (21.14). Don't forget to use conserved quantities to efficiently solve the geodesic equation.

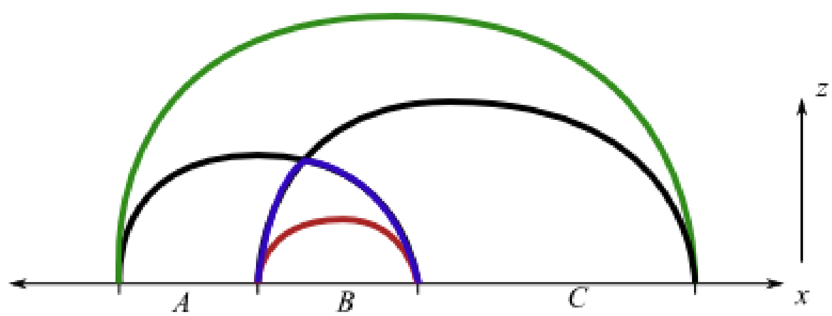
Exercise: Strips in d dimensions
Compute the holographic entanglement entropy of an infinite strip of width L_A in a d-dimensional CFT in the vacuum state. That is, with CFT coordinates $(t, x, \vec{y})$, region A is the region $x \in [-L_A/2, L_A/2]$, $t = 0$, $\vec{y}$ =anything.

21.3 Holographic proof of strong subadditivity

The proof of the strong subadditivity inequality in quantum mechanics is rather technical and tricky. The holographic proof, in static states, is easy! The statement of SSA for a tripartite system is

$$S_{ABC} + S_B \leq S_{AB} + S_{BC} \ . \tag{21.15}$$

Let's draw the various minimal-area surfaces:



(21.16)

We've picked a color scheme to reorganize the inequality a bit, so now it says

$$\text{red} + \text{green} \leq \text{blue} + \text{black} \ . \tag{21.17}$$

The fact that the curves are minimal area immediately implies

$$\text{red} \leq \text{blue}, \qquad \text{green} \leq \text{black} \tag{21.18}$$

so SSA follows.

This argument has also been extended to the time-dependent case. It is much trickier, since the extremal surfaces need not all lie in the same spatial slice.

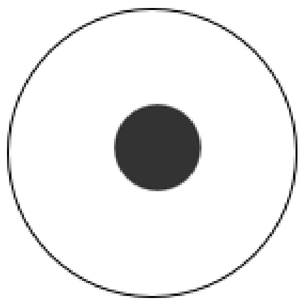
21.4 Some comments about HEE

HEE and Bekenstein-Hawking

The HEE formula is a generalization of the Bekenstein-Hawking area law for black hole entropy. To see this, let's apply the formula to a static black hole spacetime, and choose region A to be all of space. In this case the boundary condition (21.2) on the extremal surface is

$$\partial\gamma_A = \emptyset \ . \tag{21.19}$$

It is tempting to say γ is the empty set, but this would not satisfy the homology condition. A spatial slice of the black hole spacetime looks like



(21.20)

This is not simply connected, and the 'empty set' curve is not deformable to region A. So, in fact, we must choose γ_A to be the horizon itself (which is extremal). Thus

$$S_A = \frac{\text{area(horizon)}}{4G_N} \tag{21.21}$$

in agreement with Bekenstein-Hawking.

But why is this 'entanglement entropy'? Actually, it might not be. More accurately, the HEE formula computes the von Neumann entropy of the reduced density matrix, $S_A = -\operatorname{tr}\rho_A \log\rho_A$. This von Neumann entropy may or may not come from entanglement — we can't tell the difference without knowing the full system. In the black hole spacetime, the ordinary thermal entropy is the von Neumann entropy of the thermal state $\rho = e^{-\beta H}$, so the HEE formula applied to the full space gives the thermal entropy. You can also think of this as actual entanglement entropy coming from the entanglement of the CFT with the thermal double.

Some words

Entanglement entropy is a measure of how quantum information is spatially organized in a quantum state. In a general QFT, it is extremely complicated, and we do not expect any tractable simple formula. The fact that it simplifies, and becomes geometric, in holographic CFTs is a deep fact about strongly coupled systems. It means that the organization of quantum information approaches a sort of simplified, universal limit at strong coupling. How this happens and exactly how it is related to emergent geometry is an unsolved, and presumably very important, problem in current research.