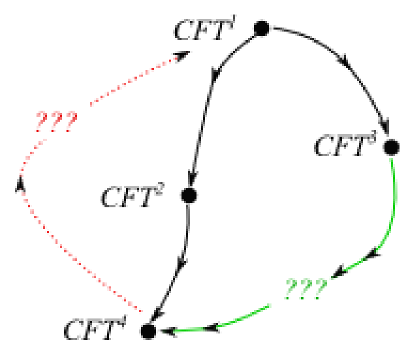
20 Entanglement Entropy and the Renormalization Group

Entanglement entropy is very difficult to actually calculate in QFT. There are only a few cases where it can be done. So what is it good for? One answer is the relation to quantum gravity, which we'll get to later. Another answer is that entanglement entropy has led to deep insights into the structure of QFT. It is a tool that is almost orthogonal to the usual tools of QFT, and can be used to prove general facts about QFT that, so far, cannot be proved using any other method. The most important example is on the irreversibility of the renormalization group in $d = 3$. We'll now take a brief detour to describe this result and the relevance of entanglement, as pioneered by Casini and Huerta. We restrict to Lorentz-invariant QFTs.

20.1 The space of QFTs

The renormalization group connects conformal field theories:*



(20.1)

Starting with CFT^1 in the UV, we deform by a relevant operator and flow down to CFT^2 or, depending on the deformation, perhaps CFT^3 in the IR. These CFTs might be free, or trivial, as in QCD, which is an RG flow between a free theory in the UV and a gapped (empty) theory in the IR. The IR fixed points may also have relevant perturbations, so we can continue the process and flow to new theories. Two natural questions are:

*Strictly speaking, it connects scale-invariant theories. It is widely suspected that scale invariant QFTs are necessarily conformal, but this is proven only in 2d and in 4d under certain assumptions.

1. Which CFTs can flow to which other CFTs? For example, can the green flow in the figure exist, connecting CFT^3 to CFT^4? Or should it flow from CFT^4 to CFT^3 instead?
2. Can there by closed cycles, connecting the IR back up to the UV like the red dotted flow in the figure?

The RG involves integrating out degrees of freedom, so it would be very strange to find closed cycles! We expect that each time we do into the IR, we reduced the number of degrees of freedom. To make this intuition precise has been a longstanding problem in quantum field theory.

20.2 How to measure degrees of freedom

To make this precise we need to define 'number of degrees of freedom.'

Free energy is no good
One 'obvious' guess fails. Let's try to measure degrees of freedom by computing the thermodynamic free energy, $\log Z$. This can be computed by the Euclidean path integral on $R^{d-1} \times S^1_\beta$. At a fixed point, dimensional analysis fixes

$$F(\beta) = -c_{therm} V_{d-1} T^d \tag{20.2}$$

where c_{therm} is a dimensionless number that we might guess counts degrees of freedom. However, c_{therm} does *not* necessarily decrease along RG flows. An example is the flow from the interacting critical point of N bosons in $d = 3$, to the Goldstone phase with $N - 1$ free bosons.

So we need a more sophisticated measure of degrees of freedom. The correct measure depends on dimension, as do known results about the irreversibility of the RG.

d=2: c-theorem
This case is the easiest and has been understood since the 80s, when Zamolodchikov

proved that the correct quantity to consider is the central charge c. Zamolodchikov's c-theorem states

$$c_{UV} \geq c_{IR} \tag{20.3}$$

in a unitary, Lorentz-invariant RG flow.

c plays many roles in a 2d CFT: It appears in the Virasoro algebra, in the trace anomaly, in the stress-tensor correlation functions, in the Casimir energy on a circle, in the thermodynamic free energy, and in the groundstate entanglement entropy. In higher dimensions, these different quantities can have different constants associated to them, so it is not obvious how to generalize (20.3) to higher dimensions. The picture that has emerged in the last few years (conjectured in even dimensions long ago by Cardy) is that the correct quantity to consider is the partition function on S^d. Exactly how this works depends on the dimension.

d=3: F-theorem

The correct measure of degrees of freedom in 3d is

$$F = -\log |Z_{S^3}| \ . \tag{20.4}$$

It can be shown that this is equal to the finite term in the entanglement entropy of a spherical region. That is, let A be a ball of radius L_A. In the vacuum state the quantity appearing in (19.5) obeys

$$\tilde{S} = F \ . \tag{20.5}$$

This quantity obeys the 'F-theorem',

$$F_{UV} \geq F_{IR} \ . \tag{20.6}$$

This was proved by Casini and Huerta using entanglement methods, described below.

d=4: a-theorem

In even dimensions, the partition function on S^d has a log divergence due from the conformal anomaly. The coefficient of this log divergence is called a:

$$\log Z_{S^4} \sim a \log \frac{R}{\epsilon_{UV}} \ . \tag{20.7}$$

The same number appears in the entanglement entropy of a spherical region. In the notation of (19.6),

$$\tilde{S} \propto a \ . \tag{20.8}$$

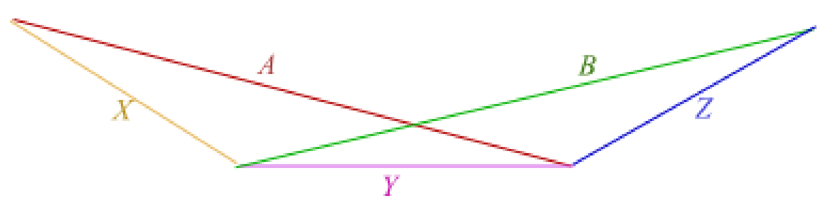
This obeys the 'a-theorem',

$$a_{UV} \geq a_{IR} \ . \tag{20.9}$$

20.3 Entanglement proof of the c-theorem

Zamolodchikov derived the c-theorem in $d = 2$ using standard QFT methods, without reference to entanglement entropy. Later, it was derived using entanglement entropy by Casini and Huerta. Their derivation is very elegant, and exemplifies how entanglement inequalities can be applied in QFT. Unlike Zamolodchikov's proof, it also generalizes to $d = 3$.

We consider any Lorentz-invariant QFT in 2d. Consider two boosted, overlapping intervals A and B, arranged as follows:



(20.10)

We have also labeled the regions X, Y, Z. All of these are spacelike regions. Comparing causal diamonds, Lorentz invariance, as discussed in section 19.2, implies

$$S_A = S_{X\cup Y} \ , \quad S_B = S_{Y\cup Z} \tag{20.11}$$

and

$$S_{A\cup B} = S_{X\cup Y\cup Z}, \qquad S_{A\cap B} = S_Y \ . \tag{20.12}$$

Now, strong subadditivity implies

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \tag{20.13}$$

i.e., (with $\cup$'s implied)

$$S_{XY} + S_{YZ} \geq S_Y + S_{XYZ}\ . \tag{20.14}$$

Parameterize the region lengths by r and R with

$$\ell(A) = \sqrt{rR}, \qquad \ell(Y) = R\ . \tag{20.15}$$

In the vacuum state, the entanglement entropy can depend only on the proper length of the region. Thus SSA becomes

$$2S(\sqrt{rR}) \geq S(R) + S(r)\ . \tag{20.16}$$

Expanding with $R = r + \epsilon$, this means

$$rS''(r) + S'(r) \leq 0 \tag{20.17}$$

or equivalently

$$C'(r) \leq 0, \qquad C(r) = rS'(r)\ . \tag{20.18}$$

(20.18) is the main technical result: the function $C(r)$ is monotonic as a function of interval size. Now for the interpretation. First, suppose our QFT is scale invariant. In this case, from (19.8), the entanglement entropy is

$$S_{cft}(r) = \frac{c}{3}\log\frac{r}{\epsilon_U V}\ . \tag{20.19}$$

Thus the Casini-Huerta C-function $C(r)$ is proportional to the central charge at a critical point,

$$C_{cft}(r) \equiv rS'(r) = \frac{c}{3}\ . \tag{20.20}$$

Now, if the QFT is not scale invariant, then it describes an RG flow between some UV CFT and some IR CFT. That is, the QFT at very short distances is equivalent to CFT_{UV}, and the QFT at very long distances is CFT_{IR}. We are interpreting the *physical* distance r as the RG scale. But we know that at the fixed points, $C(r)$ is given by the central charge,

$$C(r \to 0) = \frac{c_{UV}}{3} \ , \qquad C(r \to \infty) = \frac{c_{IR}}{3} \ . \tag{20.21}$$

Integrating the equation $C'(r) \leq 0$ from short to long distances,

$$\int_0^\infty drC'(r) \leq 0 \ . \tag{20.22}$$

This proves the c-theorem,

$$c_{UV} \geq c_{IR} \ . \tag{20.23}$$

Note that nowhere in this proof have we used the concept of a quantum field!!! We used only locality, Lorentz invariant, quantum mechanics, and unitarity (in the guise of the SSA inequality).

20.4 Entanglement proof of the F theorem

Casini and Huerta's proof of the F theorem $d = 3$ is quite similar. In this case, there is no other known way to prove that RG flows are irreversible – standard field theory methods in even dimension rely on the conformal anomaly, which does not exist in odd dimensions.

We will just briefly sketch the argument, since it is similar to $d = 2$. In a 3d CFT in vacuum,

$$S_A^{CFT} \sim \frac{r}{\epsilon_{UV}} - \tilde{S} \tag{20.24}$$

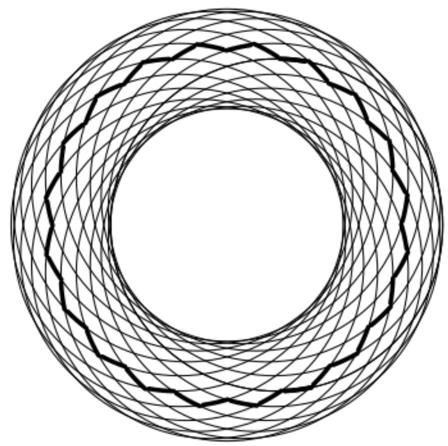
where $\tilde{S}$ is a constant, independent of r and ϵ_{UV}. Therefore a natural guess for the monotonic function is

$$F(r) = rS'(r) - S(r) \ , \tag{20.25}$$

which agrees with $\tilde{S}$ at a critical point,

$$F_{CFT} = \tilde{S} \ . \tag{20.26}$$

To use SSA, we use a more clever version of the boosted-interval setup. Two boosted balls, won't work, because the union of the causal domain of two boosted balls is not the causal domain of any ball. Instead we must arrange an infinite number of boosted regions. Projected onto a single time slice, they look like this:*



(20.27)

An argument similar to 2d implies that $F'(r) \leq 0$, which establishes the F-theorem:

$$F_{UV} \geq F_{IR} \ . \tag{20.28}$$

*Figure taken from Casini and Huerta 1202.5650.