16 Black hole thermodynamics in AdS_5

Now we return to black holes, and some of the techniques introduced in the beginning of the course.

The basic starting point is that *thermal states in CFT are dual to black holes in quantum gravity.* In fact, this is a special case of the dictionary (14.2), where we impose boundary conditions appropriate for thermal field theory. That is,

$$Z_{cft}[\phi_0; M] = Z_{grav}[\phi_0; \text{boundary} = M] \tag{16.1}$$

where we take the manifold on which the CFT lives to be

$$M = \Sigma_{d-1} \times S^1_\beta \ . \tag{16.2}$$

Here Σ_{d-1} is space. We will mostly set $\Sigma_{d-1} = S^3_\ell$, a 3-sphere of size ℓ. And S^1_β is a circle of size β. As we saw earlier in the course, the Euclidean path integral on $\Sigma_{d-1} \times S^1_\beta$ defines the finite-temperature state on Σ_{d-1}.

The meaning of the notation in (16.1) is that we calculate the gravity partition with boundary condition ϕ_0 on bulk fields, and boundary condition M on the bulk manifold. Explicitly, fields obey the usual fall-off $\phi \sim r^{-d+\Delta}\phi_0(x)$ as $r \to \infty$, and the metric itself obeys the boundary condition

$$ds^2 \to \frac{r^2}{\ell^2} dt_E^2 + \frac{\ell^2}{r^2} dr^2 + r^2 d\Omega_3^2 \ , \qquad t_E \sim t_E + \beta \ . \tag{16.3}$$

Our goal is to compute the free energy at temperature β. For this we can turn off all fields besides the metric, so $\phi_0 = 0$, and we just have the relation

$$Z_{cft}[\beta] = Z_{grav}[\beta] \ . \tag{16.4}$$

We will compute the rhs in gravity, and interpret it in CFT. The result will exhibit rich behavior, including phase transitions as a function of temperature. This will turn out to be related to confinement/deconfinement in the CFT.

16.1 Gravitational Free Energy

To compute $Z_{grav}[\beta]$, in principle, we should compute the quantum gravity path integral subject to the boundary condition (16.3). That's impossible, but in the semiclassical limit we can evaluate it approximately by expanding around classical solutions of the equations of motion. We need to find all of the classical solutions that obey this boundary condition, and evaluate their on-shell actions using

$$I_E = -\frac{1}{16\pi G_N}\int d^5x\sqrt{g}\left(R + \frac{12}{\ell^2}\right) \ . \tag{16.5}$$

If there are several solutions, then the semiclassical approximation to the path integral is

$$Z_{grav}(\beta) \approx e^{-I_E^{(1)}} + e^{-I_E^{(2)}} + \cdots \tag{16.6}$$

Each saddlpoint also comes with an infinite series of perturbative (loop) corrections but we won't worry about those, we will just evaluate the classical contributions.

There are three classical solutions (in pure gravity) obey the thermal boundary condition (16.3): small black holes, large black holes, and thermal AdS.

16.1.1 Schwarzschild-AdS

The Euclidean black hole satisfying the boundary condition (16.3) is called Schwarzschild-AdS, with metric

$$ds^2 = f dt_E^2 + \frac{dr^2}{f} + r^2 d\Omega_3^2 \ , \quad f = 1 + \frac{r^2}{\ell^2} - \frac{\mu}{r^2} \ , \tag{16.7}$$

with the thermal identification

$$t_E \sim t_E + \beta \ . \tag{16.8}$$

μ is a constant that will be related to the mass. The explicit metric on the unit 3-sphere is

$$d\Omega_3^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{16.9}$$

where ψ and θ run from 0 to π and $\phi \in [0, 2\pi)$. The horizon is the outermost solution of the equation $f(r_+) = 0$, which gives

$$r_+^2 = \frac{\ell^2}{2}\left(-1 + \sqrt{1 + \frac{4\mu}{\ell^2}}\right) \ . \tag{16.10}$$

As usual, the condition that this solution is non-singular at $r = r_+$ relates β to r_+. From the path integral point of view, this is because only smooth classical solutions are good saddlepoints; solutions with a conical defect do not satisfy the equations of motion at the defect. The conical defect trick gives

$$\beta = \frac{2\pi\ell^2 r_+}{2r_+^2 + \ell^2} \ , \tag{16.11}$$

ie

$$r_+ = \frac{\pi\ell^2}{2\beta}\left[1 \pm \sqrt{1 - \frac{2\beta^2}{\pi^2\ell^2}}\right] \ . \tag{16.12}$$

Note two things: first, there is a maximum β, ie minimum temperature,

$$\beta_{max} = \frac{\ell\pi}{\sqrt{2}} \ . \tag{16.13}$$

Second, for any given temperature β, there are *two* different black holes, corresponding to the sign choice in (16.12). Call these the 'small' (minus sign) and ''large' (plus sign) black holes. The turnover is at

$$r_* = \ell/\sqrt{2} \tag{16.14}$$

so each β allows a small black hole with $r_+ < r_*$ and a large black hole with $r_+ > r_*$.

We are working with thermal boundary conditions that fix the temperature β. So in field theory language, we are working in the canonical ensemble. Therefore we should sum over the allowed solutions; the thermodynamics will be determined by whichever has the lower free energy. We will see below that the large black hole always has lower free energy than the small black hole (but there is also a third solution, so the large black hole is not always dominate).

The behavior (16.12) is quite different from flat spacetime. In flat spacetime, larger black holes always have smaller temperature; this means ordinary Schwarzschild has negative specific heat and so is thermodynamically unstable. (It cannot be held in equilibrium with a bath, because it will absorb radiation from the bath and get *colder* the more radiation it absorbs!)

On the other hand in AdS, according to (16.12), large black holes have positive specific heat. If you make them bigger (higher energy), they get hotter. Small black holes have negative specific heat, and very small black holes $r_+ \ll \ell$ don't care about the cosmological constant at all so are just like the flat spacetime Schwarzschild solution.

On-shell action

The free energy $F = -\frac{1}{\beta}\log Z$ is computed using the on-shell Einstein action. We must be careful about the boundary terms. The full action we will use is

$$I_E = -\frac{1}{16\pi G_N}\int d^5x\sqrt{g}(R+\frac{12}{\ell^2}) + \frac{1}{8\pi G_N}\int_{r=1/\epsilon} d^4x\sqrt{\gamma}K + \int_{r=1/\epsilon} d^4x\sqrt{\gamma}L_{ct}[\gamma]\ . \tag{16.15}$$

The first term is the usual Einstein term. The second term is the Gibbons-Hawking boundary term. The last term is a counterterm; it can be any function of the intrinsic boundary geometry $\gamma_{\mu\nu}$ and will be picked to cancel divergences. Note that we are cutting off the spacetime at $r = 1/\epsilon$, we will take $\epsilon \to 0$ at the end.

Bulk term

The Einstein equation in empty spacetime implies $R = -20/\ell^2$. Thus the bulk term in (16.15) is

$$I_{bulk} = \frac{1}{2\pi G_N\ell^2}\int_{r<1/\epsilon} d^5x\sqrt{g} \tag{16.16}$$

$$= \frac{1}{2\pi G_N\ell^2}\int_0^\beta dt_E\int_{r_+}^{1/\epsilon} dr r^3\int d\Omega_3 \tag{16.17}$$

$$= \frac{\pi\beta}{4G_N\ell^2}\left[\frac{1}{\epsilon^4} - r_+^4\right]\ . \tag{16.18}$$

Boundary terms

To compute the Gibbons-Hawking-York boundary term we just plug into the definition of extrinsic curvature, and eventually find

$$I_{GHY} = -\frac{\pi\beta}{G_N \ell^2}\left[\frac{1}{\epsilon^4} + \frac{3\ell^2}{4\epsilon^2} - \frac{\mu\ell^2}{2}\right] . \tag{16.19}$$

So far our answer $I_{bulk} + I_{GHY}$ is divergent as $\epsilon \to 0$. To fix this we need to add a boundary counterterm which is a functional of γ. It turns out the only choice that makes everything finite is

$$I_{ct} = \frac{3}{8\pi G_N \ell}\int_{r=1/\epsilon} d^4x\sqrt{\gamma}\left(1 + \frac{\ell^2}{12}R[\gamma]\right) \tag{16.20}$$

where $R[\gamma]$ is the Ricci scalar for the metric γ.

Total

Plugging into the counterterm action, evaluating it, and adding everything up we find (as $\epsilon \to 0$)

$$I_E = I_{bulk} + I_{GHY} + I_{ct} = \frac{\pi^2\beta}{8G_N\ell^2}\left[r_+^2\ell^2 - r_+^4 + \frac{3\ell^4}{4}\right] . \tag{16.21}$$

This is finite, by design. This should be viewed as a function of temperature $I_E = I_E(\beta)$, so we should plug in for $r_+(\beta)$ using (16.12). If we pick the plus sign we get the action of the large black hole, and if we pick the minus sign we get the action of the small black hole. You can easily check that

$$I_E(r_+^{small}(\beta)) \geq I_E(r_+^{large}(\beta)) . \tag{16.22}$$

Thus the dominant solution, with lower free energy, is the larger black hole, at any β.

Energy and entropy

Now that we have the partition function we can use all the usual thermodynamic relations to derive things like energy and entropy. The energy is

$$E = -\partial_\beta \log Z = \frac{3\pi^2}{8G_N}\left(\mu + \frac{\ell^2}{4}\right) . \tag{16.23}$$

The first term is the relation between mass and μ. The second term is a little surprising, since it is independent of the mass! It can be interpreted as a Casimir energy induced by putting the theory on S^3. Had we chosen boundary conditions $R^3 \times S^1_\beta$, this term would be zero.

The thermodynamic entropy is

$$S = (1 - \beta\partial_\beta) \log Z = \frac{\pi^2 r_+^3}{2G_N} \tag{16.24}$$

which you can check agrees with the area law $S = \text{area}/4G_N$.

16.1.2 Thermal AdS

We're not done: there is a third solution with thermal boundary conditions (16.3). It is called *Euclidean thermal AdS* and the metric is simply

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt_E^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\Omega_3^2 \ , \tag{16.25}$$

with the identification

$$t_E \sim t_E + \beta \ . \tag{16.26}$$

This is just the metric of empty Euclidean AdS, except that we have identified the Euclidean time circle. Note that β is not related to any parameter in the metric itself, since there is no horizon r_+ in this metric. Therefore β is not fixed by any regularity condition, it is just a free parameter in this solution.

Lorentzian interpretation

We are discussing Euclidean manifolds, so let's pause to comment on the Lorentzian interpretation of all this. As discussed earlier in the course, path integrals of quantum fields on a Euclidean manifold with $t_E \sim t_E + \beta$ prepare the fields in a thermal state. For both of the Euclidean manifolds discussed here — the Euclidean black hole and thermal AdS — the path integral on the Euclidean manifold prepares a thermal state on the Lorentzian manifold. That is, the Euclidean path integral on Euclidean Schwarzschild-AdS prepares the Hartle-Hawking thermal state for fields on the the Lorentzian black hole. The Euclidean path integral on Euclidean thermal AdS prepares fields in a thermal state on ordinary Lorentzian AdS.

So in other words: In Lorentzian signature, thermal AdS is exactly the same classical solution as empty AdS, but the state of the perturbative fields is different — they are thermally populated, but their energy is small $O(\hbar)$ and does not backreact on the geometry itself.

Contractible vs non-contractible time circle

In the Euclidean black hole, t_E is the angle in polar coordinates. Together, the coordinates (r, t_E) with $r > r_+$ and $t_E \in (0, \beta)$ make a disk. The origin of the disk is a smooth point which corresponds to the Euclidean horizon.

In thermal AdS, t_E is a circle, but the circle does not contract anywhere. There is no origin. So in this geometry the coordinates (r, t_E) make a cylinder rather than a disk.

Action

The calculation of the on-shell action is similar to what we did for the black hole. Skipping the details, the answer in the end is just the Casimir term:

$$I_E^{(th)} = \frac{\pi^2 \beta}{8 G_N \ell^2} \left(\frac{3\ell^4}{4} \right) . \tag{16.27}$$

This is the free energy, which we can use to calculate the energy and entropy.

16.1.3 Hawking-Page phase transition

We found three Euclidean geometries obeying the thermal boundary condition (16.3). They are the small black hole, large black hole, and thermal AdS. The free energy of the large black hole is always smaller than that of the small black, so in understanding the phases, we can forget about the small black hole – it never dominates the canonical ensemble.

This leaves the large black hole and thermal AdS with actions

$$\begin{aligned} I_E^{(bh)}(\beta) &= \frac{\pi^2 \beta}{8 G_N \ell^2} \left[r_+^2 \ell^2 - r_+^4 + \frac{3\ell^4}{4} \right] \\ I_E^{(th)}(\beta) &= \frac{\pi^2 \beta}{8 G_N \ell^2} \left(\frac{3\ell^4}{4} \right) , \end{aligned} \tag{16.28}$$

where in $I_E^{(bh)}$ we choose the larger root for $r_+(\beta)$.

The semiclassical approximation to the gravitational path integral is the sum

$$Z_{grav}(\beta) \approx \exp(-I_E^{(bh)}) + \exp(-I_E^{(th)}) + \dots \ . \tag{16.29}$$

Each of the exponents is very large, since they are order $1/G_N$. Therefore the sum is exponentially dominated by whichever term is bigger:

$$\log Z_{grav}(\beta) \approx \max\left(-I_E^{(bh)}, \quad -I_E^{(th)}\right) \ . \tag{16.30}$$

There is a sharp (1st order) phase transition where the two solutions exchange dominance, ie at $I_E^{(th)} = I_E^{(bh)}$. Comparing the two actions, the critical temperature, and corresponding black hole radius, for this phase transition is

$$\beta_{crit} = \frac{2\pi\ell}{3} \ , \qquad r_+^{crit} = \ell \ . \tag{16.31}$$

The low-temperature phase is thermal AdS; the high temperature phase is the black hole. This phase transition is called the Hawking-Page transition and was discovered well before AdS/CFT. The story is qualitatively the same in any number of dimensions, AdS_{d+1} (with a few differences in AdS_3).

Entropy

The entropy of thermal AdS is zero. We can see this either by noting there is no horizon, or computing $(1-\beta\partial_\beta)\log Z = 0$. Actually, it is not exactly zero, since we have only computed the semiclassical term. There are quantum corrections, and the true entropy is $O(G_N^0)$ from the one-loop contribution (*i.e.,* determinant of gravitons matter fields in AdS).

Thus full thermal entropy $S(\beta)$, accounting for the phase transition, is $O(G_N^0)$ at low temperatures and then suddenly jumps to a very large number $O(1/G_N)$ at β_{crit}. In the microcanonical ensemble, where we view this as a function of energy $S(E)$, the entropy is related to the density of states

$$S(E) = \log \rho(E) \ . \tag{16.32}$$

The Hawking-Page transition indicates that theories with a semiclassical gravity description must have a small number of states at low energy, but an enormous number of states at high energy, with a sharp transition.

16.1.4 Large volume limit

We have been computing the free energy at temperature β for the theory on the space S^3_ℓ, a 3-sphere of size ℓ. In fact since we are in conformal field theory, only the ratio ℓ/β is meaningful, as this is the only dimensionful parameter. In other words the only parameter is ℓT. Going to high temperatures is therefore the same as going to large ℓ.

If we are interested in the theory on R^3 we can take $\ell \to \infty$. This is the same as taking the temperature $T \to \infty$. In this limit, from (16.28), the free energy becomes (with $I_E = \beta F$)

$$F \approx -\left(\frac{\ell^2}{G_N}\right)\ell^3\pi^6 T^4 \ , \tag{16.33}$$

i.e.,

$$F \sim -\left(\frac{\ell}{\ell_P}\right)^3 V T^4 \tag{16.34}$$

where V is the volume of the system. Up to the prefactor, we could have guessed this answer from dimensional analysis. In a conformal field theory on R^{d-1} the only dimensionful scale is the temperature, and F must be proportional to volume, so conformal invariance implies $F \sim VT^d$.

Note that the theory on the plane has only one phase: the black hole phase. There is no Hawking-Page transition on R^3. It is essentially always in the high temperature phase.

16.2 Confinement in CFT

Any CFT with a semiclassical holographic dual must share the same thermodynamics, summarized by (16.30). What does this mean about the CFT? The microcanonical entropy tells us about the spectrum: we must have an enormous number of degrees of freedom to reproduce the high-energy density of states. However we must have a small number of states at low energies. This sounds like confinement! In a confining $SU(N)$ gauge theory, in the confining phase, the physical states are color singlet hadrons, and the free energy is $F = O(1)$. In a deconfined phase, the states are gluons, and the free energy is $F \sim O(N^2)$. This agrees with a our results above (after subtracting off the contribution from the Casimir energy, which is a temperature-independent contribution to F and does not affect the entropy). The black hole phase is like the deconfined phase, and the confined phase is like thermal AdS.

This analogy comes with some caveats, so let's compare and contrast QCD with a holographic theory like $\mathcal{N} = 4$ Super-Yang-Mills. In QCD, there is a confinement/deconfinement phase transition in infinite volume, ie for the theory on R^3. It is confining at low temperatures and deconfined at high temperatures. According to our gravity results, $\mathcal{N} = 4$ Super-Yang-Mills (at strong coupling) does *not* have a confining phase on R^3. CFT's on R^3 cannot have phase transitions, because the temperature can always be rescaled (unless there is some other parameter turned on, like a chemical potential). So in fact $\mathcal{N} = 4$ SYM is not a confining gauge theory in the same sense as QCD.

In gravity, the phase transition is on S^3. Normally it is not possible to have a phase transition in finite volume – with a finite number of degrees of freedom, the free energy is an analytic function of β, and we get sharp phase transitions only in the thermodynamic limit. However this is possible in gravity because of the large-N limit. The same statements are true of $\mathcal{N} = 4$ SYM on a sphere: it has something like a confinement/deconfinement transition on the sphere, in the $N \to \infty$ limit. It is not the same sort of confinement as QCD, which comes from dynamics in a very complicated way. In gauge theory on the sphere, we get 'kinematic confinement' just from the Gauss law constraint, which does not allow charges states on a compact space. Therefore the physical states on a sphere cannot have any net color. This is what 'confines' the theory so that there are $O(1)$ physical states at low energies.

Temporal Wilson loop

(This will be very brief; see Kirtsis for more discussion.)

Define the temporal Wilson loop $W = \text{ Tr } \exp \oint A$ where the integral is over a worldline going around the thermal time circle. This is an order parameter for the deconfinement transition:

$$\langle W \rangle \neq 0 \quad \Rightarrow \quad \text{deconfinement} \quad \Rightarrow \quad \text{black hole phase} \tag{16.35}$$

$$\langle W \rangle = 0 \quad \Rightarrow \quad \text{confinement} \quad \Rightarrow \quad \text{thermal AdS phase} \tag{16.36}$$

($\langle W \rangle \neq 0$ actually breaks a symmetry: the center of the gauge group, $Z_N \in SU(N)$.) A rough explanation is that you can think of the temporal Wilson loop as a free quark. If a free quark has finite energy, then you get $\langle W \rangle \neq 0$, but if the free quark has infinite energy then $\langle W \rangle = 0$.

To compute a Wilson loop in AdS/CFT from the gravity side, the rule is to find a string worldsheet ending on the Wilson line and extending into the bulk. This classical string diagram computes the leading contribution to the Wilson loop at large N.

In the Euclidean black hole, since (t_E, r) make a disk, it is easy to find a string worldsheet ending on this Wilson line. In thermal AdS, however, since the thermal circle is not contractible — ie (t_E, r) make a cylinder — you cannot find such a string worldsheet, and the Wilson line vanishes.

Thus the deconfined phase is the phase with a contractible thermal circle in the dual geometry, and the confined phase has a non-contractible thermal circle.

16.3 Free energy at weak and strong coupling

So can we calculate the free energy of $\mathcal{N} = 4$ SYM, and compare to (16.30)? Unfortunately, no. The gravity calculation is dual to $\mathcal{N} = 4$ SYM at very strong 't Hooft coupling, $\lambda \equiv g_{YM}^2 N \to \infty$. The free energy is not protected by supersymmetry, and it is unknown how to do this calculate in gauge theory at strong coupling.

But we can do the CFT calculation at weak coupling. The free energy of a weakly coupled QFT is just a 1-loop calculation, ie a determinant for each of the fields. This calculation has been done. It agrees qualitatively, but not quantitatively, with the gravity calculation. For example, after translating all the parameters of CFT to gravity parameters, the free energy of free $\mathcal{N} = 4$ SYM on R^3 is

$$F_{free} = \frac{4}{3} F_{gravity} \ , \tag{16.37}$$

where $F_{gravity}$ is given in (16.33). This famous factor of 4/3 is not a contradiction. It just means that the free energy at strong coupling is different from the free energy at weak coupling.

In principle, or perhaps on a lattice, the free energy is some function of the coupling,

$$F = -f(\lambda) \frac{\pi^2}{6} N^2 V T^4 \ . \tag{16.38}$$

We know the behavior of $f(\lambda)$ as $\lambda \to 0$ and as $\lambda \to \infty$. We only described the leading terms above, but it is also possible to calculate corrections. On the CFT side, corrections come from higher loops. On the gravity side, corrections come from including higher curvature (stringy) contributions to the classical action. (In principle we could also ask about $1/N$ corrections, which would require quantum calculations on the gravity side, but we'll restrict to the leading-N behavior.) These corrections have been calculated and lead to

$$\text{gravity:} \quad f(\lambda) = \frac{3}{4} + \frac{45}{32} \frac{\zeta(3)}{\lambda^{3/2}} + \cdots \quad \text{as} \quad \lambda \to \infty \tag{16.39}$$

and

$$\text{CFT:} \quad f(\lambda) = 1 - \frac{3}{2\pi^2} \lambda + \cdots \quad \text{as} \quad \lambda \to 0 \tag{16.40}$$

Evidently the corrections are heading the right direction, but the full function $f(\lambda)$ is unknown.

Exercise: Hawking-Page in Three Dimensions

Recall the metric of the Euclidean BTZ black hole in AdS_3,

$$ds^2 = \ell^2 \left[(r^2 - 8M) dt_E^2 + \frac{dr^2}{r^2 - 8M} + r^2 d\phi^2 \right] \ . \tag{16.41}$$

In a previous exercise, you computed the on-shell Euclidean action of this black hole. The answer, including all boundary terms and counterterms, is

$$S_E^{(bh)}(\beta) = -\frac{\pi^2 c}{3\beta} \tag{16.42}$$

where

$$c = \frac{3\ell}{2G_N} \ . \tag{16.43}$$

1. Like in AdS_5, there is also a thermal AdS solution with the same boundary condition.[71] We will use a trick to compute its action. The trick is to note that (16.41) is a solid torus with boundary $S^1_{2\pi\ell} \times S^1_\beta$. (The subscript is the circumference of the S^1). The thermal circle S^1_β is 'filled in' to make the solid torus.

   Thermal AdS_3 is a solid torus where instead the *other* circle $S^1_{2\pi\ell}$ is 'filled in' to make solid torus. Argue that this implies

   $$S_E^{(th)}(\beta) = S_E^{(bh)}\left(\frac{4\pi^2}{\beta}\right) = -\frac{c\beta}{12} \ . \tag{16.44}$$

   Comment: This is a special case of a *modular transformation*. It is a 'large'[72] conformal transformation acting on a torus, which roughly speaking relates a fat torus to a skinny torus.

2. Sketch a plot of the free energies $F^{(bh)}$ and $F^{(th)}$. Find the critical temperature β_{crit} of the Hawking-Page phase transition, and write $\log Z(\beta)$ as a piecewise function.

3. Find the thermodynamic entropy $S(\beta)$ for all $\beta > 0$.

4. Find the energy $E(\beta)$ for all $\beta > 0$.

5. Use part (4) in part (3) to find the entropy in the microcanonical ensemble, $S(E)$. (Be careful about what ranges of E your formulas apply to; in particular you cannot find $S(E)$ for all E by this method.)

[71] Unlike AdS_5, there is only one black hole with temperature β.

[72] *i.e.,* not continuously connected to the identity

6. Interpret your results in terms of the density of states in a 2d CFT dual to 3d gravity.