15 Correlation Functions in AdS/CFT

As a first application of (14.2), we will use the gravity side to derive the correlators of a conformal field theory. First we'll start with a purely QFT discussion of correlators in a theory with conformal invariance, then reproduce these results from gravity.

By the way, you've already seen one example of CFT correlators compute from gravity: the absorption cross section calculation. That was related to a CFT correlator at finite temperature. In this section are deriving correlators in the vacuum state, *i.e.,* empty AdS.

15.1 Vacuum correlation functions in CFT

This will be a brief introduction to CFT. For details, see: Polchinski's String Theory book; Kiritsis's String Theory book; or the big yellow CFT book by Di Francesco et al.

The group of conformal symmetries of R^d is $SO(d+1,1)$. In Lorentz signature, the conformal symmetries of $R^{d-1,1}$ are $SO(d,2)$. The generators of $SO(d,2)$ are

$$\begin{aligned}
P_\mu &= -i\partial_\mu \\
L_{\mu\nu} &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
D &= -ix^\mu\partial_\mu \\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)
\end{aligned} \tag{15.1}$$

The the first two lines are translations, rotations, and boosts; these generate the Poincare group (which is 10-dimensional in $d = 4$). The 3rd line is the dilatation, or scale generator, since under $x^\mu \to x^\mu + i\epsilon D^\mu$, the coordinate is just rescaled, $x^\mu \to x^\mu(1+\epsilon)$. The last line is called the special conformal transformation.

One way to derive (15.1) is to find the conformal Killing vectors of Minkowski space. These are defined to be vectors V^μ obeying

$$\mathcal{L}_V \eta_{\mu\nu} = f(x)\eta_{\mu\nu} \ , \tag{15.2}$$

where f is any function. This is the infinitessimal version of the definition of a conformal symmetry, which maps $ds^2 \to e^{\Omega(x)} ds^2$.

Operators in a CFT can be organized under representations of the conformal group. We define *primary operators* to obey[70]

$$
\begin{aligned}
[D, O(0)] &= -i\Delta O(0) \qquad (15.3)\\
[K_\mu, O(0)] &= 0\ . \qquad (15.4)
\end{aligned}
$$

The dilatation eigenvalue Δ is called the scaling dimension of O. The second condition is like a highest weight condition. We can build the full representation by acting on $O(x)$ with the conformal generators, so for example $\partial_m u O(x)$ is a *descendant operator*.

The finite version of of the D commutator says that under a rescaling $x \to \lambda x$, we have $O(x) \to \lambda^\Delta O(\lambda x)$. More generally, primaries obey

$$
O'(x') = \left| \det \frac{\partial x'^\mu}{\partial x^\nu} \right|^{-\Delta/d} O(x)\ . \qquad (15.5)
$$

For a correlation function of n primaries this implies

$$
\langle O_1(\lambda x_1) \cdots O_n(\lambda x_n) \rangle = \lambda^{-\Delta_1 - \Delta_2 - \cdots - \Delta_n} \langle O_1(x_1) \cdots O_n(x_n) \rangle\ . \qquad (15.6)
$$

The special conformal transformations also impose requirements on correlators. It turns out that all the conformal generators together completely fix the 2 and 3-point functions of a CFT, up to overall factors. The two-point function of equal-weight fields is

$$
\langle O_1(x_1) O_2(x_2) \rangle = \frac{c_{12}}{|x_1 - x_2|^{2\Delta}} \qquad (\Delta_1 = \Delta_2 \equiv \Delta) \qquad (15.7)
$$

and it must vanish if $\Delta_1 \neq \Delta_2$. The number c_{12} can be rescaled by rescaling our normalization of the operators. Often we pick an orthonormal basis of primary operators, so that $c_{ij} = \delta_{ij}$.

[70] This is for scalars. Operators with spin would also have the usual rule for action by the Lorentz group, $[L_{\mu\nu}, O(0)] = \Sigma_{\mu\nu} O(0)$.

Similarly, the only 3-point function allowed by conformal invariance is

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \ , \tag{15.8}$$

where

$$x_{ij} \equiv x_i - x_j \ . \tag{15.9}$$

The number c_{ijk}, called an OPE coefficient (for operator product expansion), is a real physical prediction of the theory, since we've already fixed normalizations via the 2-point function.

In fact, the set of scaling dimensions Δ_i and the OPE coefficients c_{ijk} are *all* the data of a CFT. This is because higher correlators can be computed, at least in principle, by sewing together 3-point functions and summing over intermediate states.

The 4-point function is not completely fixed by conformal symmetry, but it is highly constrained. With equal external weights $\Delta_{1,2,3,4} = \Delta$, the most general form of the 4-point correlator is

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = |x_{12}|^{-2\Delta}|x_{34}|^{-2\Delta}F(u,v) \tag{15.10}$$

where F is an arbitrary function of the *conformal cross ratios*,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ . \tag{15.11}$$

15.2 CFT Correlators from AdS Field Theory

We assume a strongly coupled, large-N CFT with a semiclassical holographic dual. This is a limit where the gravity theory is weakly coupled, $G_N \ll \ell$, and higher curvature corrections can be neglected, $\ell_{string} \ll \ell$ (here ℓ is the AdS radius). According to the GKPW dictionary (14.2), we can compute the generating function of CFT correlators on the gravity side by

$$Z_{cft}[\phi_0] \equiv \langle e^{-\int \phi_0 O}\rangle_{CFT} \tag{15.12}$$

$$\approx \exp\left(-S_{grav} + O(G_N^0) + O(\frac{\ell_{string}}{\ell_{AdS}})\right) \tag{15.13}$$

where S_{grav} is the on-shell action for gravity subject to the boundary condition

$$\phi \to z^{-\Delta+d}\phi_0(x) \tag{15.14}$$

as we approach the AdS boundary $z \to 0$.

So to compute CFT correlators, we need to understand how to compute the classical action in AdS as a functional of the boundary conditions.

This material is explained very clearly in many places, so I will not repeat it here. I recommend reading Witten's original paper on the subject, where 'Witten diagrams' were introduced [hep-th/9802150]. In class, I followed, almost exactly Kiritsis's String Theory in a Nutshell sections 13.8.1 and 13.8.2. Read those sections before continuing!

15.3 Quantum corrections

So far we only used the classical theory on the gravity side. (Though on the CFT side, this is a strongly coupled QFT calculation which is not at all classical!) What happens when we include loop corrections in the gravity? The gravitational loop expansion is organized into powers of G_N. The classical term is $\sim 1/G_N$. If we compute Witten diagrams with loops, then we find an expansion in G_N.

On the CFT side, this is an expansion in $1/N_{dof}$, since recall the dictionary $\ell^{d-1}/G_N \sim N_{dof}$.

This implies something very special about CFTs with a semiclassical holographic dual: These CFTs, although strongly coupled, have a meaningful expansion in $1/N_{dof}$. Defining $N_{dof} = N^2$ (since this notation holds in $SU(N)$ gauge theory), this can be restated as the fact that connected correlation functions are suppressed. That is, if we normalize our operators by setting

$$\langle OO \rangle \sim 1 \tag{15.15}$$

(with the appropriate factors of x suppressed), then the 3-point function is suppressed,

$$\langle OOO \rangle \sim \frac{1}{N} \tag{15.16}$$

and higher-point functions are dominated by their connected piece:

$$\langle OOOO\rangle \sim \langle OO\rangle\langle OO\rangle + O(1/N^2) \ . \tag{15.17}$$

The explicit theories we know of with this sort of behavior are large-N gauge theories. These have been studied for a long time, starting with a beautiful paper by 't Hooft in the 70s where he showed that the Feynman diagrams of $SU(N)$ gauge theory in the large-N limit naturally reorganize themselves into something that looks roughly like a string theory. I will not cover this, but I highly recommend you read about it in section 13.1 of Kiritsis, or the big AdS/CFT review [hep-th/9905111].

Another consequence of the weak coupling constant G_N on the gravity side is that gravity has an approximate Fock space. That is, if we have a weakly coupled scalar field on the gravity side, then we can construct 1-particle states, 2-particles states, etc, by acting with creation operators. On the CFT side, this means for example that if we have a primary O_1 of dimension Δ_1, and another primary O_2 of dimension Δ_2, then there is a third primary O_{1+2} of dimension

$$\Delta_{O_{1+2}} \approx \Delta_1 + \Delta_2 + O(1/N) \ . \tag{15.18}$$

This is very special; it does not happen in general CFT, where states are just a some strongly coupled mess and there is no way to 'add' some stuff to other stuff without getting large corrections to the conserved charges from the strong interactions.

Following the gauge theory language, the operators dual to single bulk fields are called 'single-trace operators', and the operators like O_{1+2} are called 'multi-trace operators' and usually just denoted by the product $O_1 O_2$ (or more complicated things like $O_1 \Box^n \partial_{\mu_1\cdots\mu_\ell} O_2$).

In words, (15.18) says that in a CFTs with a semiclassical holographic dual, low dimension operators have 'small anomalous dimensions.' I've restricted this statement to low-dimension operators because these are the operators dual to bulk fields; high dimension operators are dual to non-perturbative stuff like black hole microstates.