13 Absorption cross section from the dual CFT

Now we will reproduce the absorption cross section (12.21) using holography. This requires introducing some elements of conformal field theory. We will be more systematic about CFT and about the AdS/CFT correspondence later, for now we are just going to work this example in full detail as an illustration.

13.1 Brief Introduction to 2d CFT

(*References:* Polchinski's String Theory Ch 2 is a brief introduction. For a more detailed systematic introduction to 2d CFT, see chapters 4-6 (especially chapter 5) of the (highly recommended!) book *Conformal Field Theory* by Di Francesco et al.

Consider a 2d QFT on the Euclidean plane $\mathbf{R}^2$, with coordinates x_1 and x_2. It is very convenient to use the complex coordinates

$$z = x_1 + ix_2, \qquad \bar{z} = x_1 - ix_2 \ . \tag{13.1}$$

We take the flat metric on the plane,

$$ds^2 = (dx_1)^2 + (dx_2)^2 = dz d\bar{z} \ . \tag{13.2}$$

2d Conformal Transformations

A *conformal transformation* is a coordinate transformation that leaves the metric unchanged, up to an overall rescaling:

$$ds^2 = dz d\bar{z} \to e^{\sigma(w,\bar{w})} dw d\bar{w} \ . \tag{13.3}$$

First we want to find what type of coordinate changes have this special property. To this end, consider an arbitrary coordinate change $z = f(w, \bar{w}), \bar{z} = \bar{f}(w, \bar{w})$ where $\bar{f}$ is the complex conjugate of f. The metric in $(w, \bar{w})$ coordinates is

$$ds^2 = \left(\frac{\partial f}{\partial w} dw + \frac{\partial f}{\partial \bar{w}} d\bar{w}\right)\left(\frac{\partial \bar{f}}{\partial w} dw + \frac{\partial \bar{f}}{\partial \bar{w}} d\bar{w}\right) \ . \tag{13.4}$$

For this to have the form (13.3) we must impose

$$\frac{\partial f}{\partial w}\frac{\partial \bar{f}}{\partial w} = \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{w}} = 0 \ . \tag{13.5}$$

This is equivalent to the condition that f is a holomorphic function,

$$f = f(w), \qquad \bar{f} = \bar{f}(\bar{w}) \ . \tag{13.6}$$

Thus conformal transformations in two dimensions are equivalent to holomorphic coordinate changes. The conformal group is the group of holomorphic maps. This is infinite dimensional, since you need an infinite number of parameters to specify a whole function. Note that this is not the case in higher dimensions; the conformal group in $d > 2$ dimensions is the finite-dimensional group $SO(d, 2)$.

Mapping the plane to the cylinder

A very important conformal transformation is the mapping of the z-plane to the w-cylinder. The mapping is

$$z = e^{-iw/R} \ , \qquad \bar{z} = e^{i\bar{w}/R} \ . \tag{13.7}$$

The w coordinate labels a cylinder, since if we take $w \to w + 2\pi R$ we get back to where we started. That is, w is identified,

$$w \sim w + 2\pi R \ . \tag{13.8}$$

This circle is a circle of constant magnitude on the z plane. (Draw the pictures for yourself.) If we split w into real coordinates,

$$w = \sigma_1 + i\sigma_2 \ , \tag{13.9}$$

then $\sigma_1 \sim \sigma_1 + 2\pi R$ is the circle and σ_2 is infinite. Negative values of σ_2 correspond to small circles on the z plane, and larger values of σ_2 correspond to increasingly larger circles on the z plane.

Classical CFT

At the classical level, a QFT has conformal symmetry if the action is invariant under conformal transformations. For example consider the action of a free massless scalar

$$S = \int d^2 z \partial\phi \bar{\partial}\phi \tag{13.10}$$

where we use the notation

$$\partial = \partial_z, \qquad \bar{\partial} = \partial_{\bar{z}} \ . \tag{13.11}$$

Perform the infinitesimal coordinate change $z \to w(z)$ and it is easy to check that the Jacobian in the measure cancels the factors that show up from $\partial\phi = \frac{dw}{dz}\partial_w\phi$. On the other hand the free massive scalar is not conformally invariant. This illustrates a general feature of conformal field theories: they do not have any dimensionful parameters. Dimensionful parameters set a scale and therefore are not compatible with the scale transformation $z \to \lambda z$, which is part of the conformal group (in any number of dimensions).

Quantum CFT

Classical conformal invariance does not necessarily imply quantum conformal invariance. This is familiar from QCD (setting all quark masses to zero) — this theory is classically scale invariant, but to define the quantum theory we must introduce a regulator, and this leads to the dimensionful QCD scale Λ_{QCD} with important physical consequence (like confinement), so QCD is certainly not scale-invariant or conformally invariant at the quantum level. From now on when we say 'CFT' we mean at the quantum level.

Primary operators

The local operators of a CFT must transform covariantly. *Primary operators*[61] transform with a simple rescaling,

$$O'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} O(z,\bar{z}) \tag{13.12}$$

[61] These are often called primary *fields*. The names are interchangeable. But remember that in CFT, a 'field' is not necessarily a fundamental field that appears in the Lagrangian and is integrated over in the functional integral. For example in the free massless scalar, $\partial\phi$ is a primary field.

where $(h, \bar{h})$ are called the *conformal weights*. Another common notation is

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h} \ , \tag{13.13}$$

where Δ is the *scaling dimension* and s is the *helicity*. Δ is the weight under a constant rescaling $(x_1, x_2) \to (\lambda x_1, \lambda x_2)$, ie under

$$\delta z = \lambda z, \qquad \delta \bar{z} = \lambda \bar{z} \tag{13.14}$$

the operator transforms with a factor of $\lambda^{-\Delta}$. s is the helicity because it is the weight under a rotation $(x_1, x_2) \to (x_1 - \lambda x_2, x_2 + \lambda x_1)$, ie under

$$\delta z = \lambda z, \qquad \delta \bar{z} = -\lambda \bar{z} \tag{13.15}$$

the operator transforms with a factor of λ^{-s}. The absolute value $|s| = |h - \bar{h}|$ is the *spin* of the operator. (This is just the usual definition of spin, so for example in free field theory it corresponds to the number of Lorentz indices on a field.

Descendant operators are operators that you get from primaries by acting with conformal transformations. For example, $\partial O(z, \bar{z})$ is a descendant of $O(z, \bar{z})$. The transformation law for descendants is more complicated than (13.12) but is completely fixed by symmetry.

All local operators in a CFT are either *primary* or *descendant*. This ensures that correlation functions transform covariantly under the conformal group. For example, the 2-point function on the plane must have the form

$$\langle O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{13.16}$$

where

$$h = h_1 = h_2, \qquad \bar{h} = \bar{h}_1 = \bar{h}_2 \ . \tag{13.17}$$

C_{12} is a constant, related to the normalization of the field. The two-point function vanishes if the conformal weights of the two fields are different.

The path-integral definition of $\langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)\rangle$ in (13.16) is (up to normalization)

$$\langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)\rangle = \int D\Phi O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)e^{-S[\Phi]} \tag{13.18}$$

where Φ stands for the fundamental fields of the theory.[62] Recall from our discussion of Euclidean path integrals that the path integral on a half-plane prepares the vacuum state. Therefore in operator langauge,

$$\langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)\rangle = {}_{line}\langle 0|O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)|0\rangle_{line} \ , \tag{13.19}$$

where $|0\rangle_{line}$ is the vacuum state of the theory on an infinite line (which you can think of as the Im $z=0$ axis).

13.2 2d CFT at finite temperature

Remember from our discussion of Euclidean path integrals that QFT at finite temperature in Lorentzian signature is related to Euclidean QFT on a cylinder, with periodic imaginary time. Now we will see this relation very explicitly in CFT.

Mapping to the cylinder via $w = iR\log z$, and applying the transformation law (13.12) to (13.16), we can easily find the cylinder correlation function

$$\langle O_{cyl}(w_1,\bar{w}_1)O_{cyl}(w_2,\bar{w}_2)\rangle \sim \frac{R^{-2h}}{\sin\left(\frac{w_1-w_2}{2R}\right)^{2h}}\frac{R^{-2\bar{h}}}{\sin\left(\frac{\bar{w}_1-\bar{w}_2}{2R}\right)^{2\bar{h}}} \ . \tag{13.20}$$

(The 'cyl' subscript is usually dropped, so functions of w are just assumed to be cylinder operators.)

Exercise: Conformally invariant 2-point functions

(a) Prove (13.16).

[62] Often we do not have a Lagrangian for a CFT, and there is no useful notion of the 'fundamental' fields. However, path integral manipulations are still useful. Even in non-Lagrangian theories we never get into trouble by pretending that there are some fundamental fields defining the functional integral.

(b) Derive (13.20), including the missing coefficient.

Note two things about this correlator: First, it is invariant under the cylinder periodicity $w_1 \sim w_1 + 2\pi R$.[63] Second, it has the same short-distance singularity as the plane correlator (13.20), *i.e.,*

$$\langle O_{cyl}(w_1, \bar{w}_1) O_{cyl}(w_2, \bar{w}_2) \rangle = \frac{C_{12}}{(w_1 - w_2)^{2h} (\bar{w}_1 - \bar{w}_2)^{2\bar{h}}} \quad \text{as} \quad w_1 \to w_2 \ . \tag{13.21}$$

This is always true in QFT: the short-distance behavior is fixed by vacuum correlation functions. (In fact these two conditions fix the function (13.20) uniquely, assuming some behavior at infinity, so we do not even strictly need the exponential mapping to derive (13.20).)

From the Lorentzian point of view, the cylinder correlator (13.20) can be interpreted different ways. To go to Lorentzian signature, write $w = \sigma_1 + i\sigma_2$ where $\sigma_{1,2}$ are real coordinates. If we think of σ_2 as 'time', then the Wick rotation to Lorentzian signature is $\sigma_2 = it$. In this case the circle coordinate σ_1 remains a circle in Lorentzian signature, so this Wick rotation gives the Lorentzian theory on the Lorentzian cylinder $S^1 \times$ Time. This Wick rotation has nothing to do with finite temperature.

To get the finite temperature theory, we instead Wick-rotate by setting $\sigma_1 = it$. Thus

$$w \to i(t + x), \qquad \bar{w} = i(t - x) \ . \tag{13.22}$$

(Note that in Lorentzian signature, w and $\bar{w}$ are no longer complex conjugates.) This means that the the theory is periodic in imaginary time $t \sim t + 2\pi i R$. Comparing to the finite-temperature periodicity $t \sim t + i\beta$ with $\beta = T^{-1}$, we see that our Euclidean CFT is related to a finite-temperature CFT at temperature

$$\beta = T^{-1} = 2\pi R \ . \tag{13.23}$$

[63] There are some subtleties with branch cuts making this statement that we'll ignore for now, and it relies on the fact that $(-1)^{2(h-\bar{h})} = 1$ since operators must have integer or half-integer spin.

From (13.20), this means the finite-temperature Lorentzian correlator in CFT is[64]

$$\begin{aligned} G_\beta(t-i\epsilon,x) &= \text{Tr } e^{-\beta H} O(t-i\epsilon,x) O(0,0) \\ &\sim (-1)^{h+\bar{h}} \frac{(\pi T)^{2h}}{\sinh(\pi T(t+x))^{2h}} \frac{(\pi T)^{2\bar{h}}}{\sinh(\pi T(t-x))^{2\bar{h}}} \ . \end{aligned} \tag{13.24}$$

13.3 Derivation of the absorption cross section

We now return to the derivation of the absorption cross-section (12.21). Recall that we scattered a low-energy quantum from the near-extremal black string. The near horizon region relevant to this calculation was a BTZ black hole in AdS_3 (times S^3). We will set $T_L = T_R = T_H$ for simplicity, which corresponds to setting the parameter $\sigma = 0$ in the black string metric. From the point of view of the near-horizon, this sets the angular momentum of the BTZ black hole to zero.

In the gravity calculation (12.21), we found

$$\sigma_{abs} \sim \coth\left(\frac{\omega}{4T_H}\right) \ . \tag{13.25}$$

Now the claim is as follows:

We can replace the near-horizon geometry by 1+1d CFT at temperature $T = T_H$, living on a fictitious 'membrane' at the boundary of AdS_3.

This boundary was the matching location in our gravity calculation, ie some value of r in the range $r_0 \ll r \ll r_{1,5}$.

Which CFT is it?

We will only match the temperature dependence. The overall factor can also be matched by this method, up to a constant. We will not need to specify *which* CFT we are actually considering, we will just need some general properties of the CFT like the value of the temperature, and the existence of an operator with certain conformal weights. The microscopic definition of the *particular* CFT depends the particular

[64] Setting a convenient normalization, and introducing an $i\epsilon$ to keep track of operator ordering. Recall that the finite-temperature correlator is defined by ordering in Euclidean-time, so this sets the order of the operators in the trace as written.

theory of quantum gravity. The only known microscopic CFTs are the ones coming from string theory, since that is our only candidate theory of quantum gravity, but in principle there could be other CFTs corresponding to other UV completions of gravity. In the string theory examples, where the microscopic definition of the CFT is known, it is also possible to match the coefficient in the absorption calculation and it comes out correctly.

The interaction term

We want to scatter a scalar field against the CFT. We will assume that the bulk scalar field χ couples to a CFT operator O, thus adding to the CFT an interaction term

$$S_{int} = \int dtdx O(t,x)\chi(t,x,r=0) \ . \tag{13.26}$$

In this expression O is a CFT operator and $\chi(r=0)$ — the value of the bulk field at the fictitious membrane where the CFT lives — is treated as a classical source. We will assume that the space direction in the CFT is unwrapped, so we call it $x \in (-\infty, \infty)$ (previously called ϕ), though the S^1 version can also be done with some extra assumptions about the CFT. We also assume the source couples weakly to the CFT so that the interaction term (13.26) can be treated perturbatively.

Absorption rate

When we computed the absorption cross section, we assume

$$\chi = e^{-i\omega t}R(r) \ , \tag{13.27}$$

so $S_{int} \propto \int dtdx O(t,x)e^{-i\omega t}$. The transition amplitude from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by Fermi's Golden Rule, as the matrix element of the interaction Hamiltonian

$$\mathcal{M}_{i\to f} \sim \langle f| \int dtdx O(t,x)e^{-i\omega t}|i\rangle \ . \tag{13.28}$$

The total absorption rate at temperature β is computed by summing this over final states, and averaging over initial states with a thermal ensemble,

$$\Gamma_{abs} \sim \sum_{i,f} e^{-\beta E_i} \int dt_1 dx_1 dt_2 dx_2 e^{-i\omega(t_1-t_2)} \langle i|O(t_2,x_2)|f\rangle\langle f|O(t_1,x_1)|i\rangle \tag{13.29}$$

The sum over $|f\rangle$ is just the identity, so up to an overall factor of volume (which you can think of as the momentum-conserving delta function $\delta(0)$),

$$\Gamma_{abs} \sim \int dtdxe^{-i\omega t} \sum_i e^{-\beta E_i} \langle i|O(t,x)O(0,0)|i\rangle \ . \tag{13.30}$$

This sum is the definition of the thermal 2-point function,

$$\Gamma_{abs} \sim \int dtdxG_\beta(t - i\epsilon, x)e^{-i\omega t} \tag{13.31}$$

This thermal correlator was calculated in (13.24). To take the Fourier transform, use the integral

$$\int dye^{-i\omega y}(-1)^h \left(\frac{\pi T}{\sinh\left[\pi T(y \pm i\epsilon)\right]}\right)^{2h} = \frac{(2\pi T)^{2h-1}}{\Gamma(2h)} e^{\pm\omega/2T} \left|\Gamma(h + i\frac{\omega}{2\pi T})\right|^2 . \tag{13.32}$$

First take the Fourier transform assuming indepdendent left and right momenta

$$G_\beta(\omega_L, \omega_R) \quad = \quad (-1)^{h+\bar{h}} \int dtdxe^{-i\omega_L(t+x)-i\omega_R(t-x)} \frac{(\pi T)^{2h}}{\sinh(\pi T(t+x))^{2h}} \frac{(\pi T)^{2\bar{h}}}{\sinh(\pi T(t-x))^{2\bar{h}}}$$

and then set $\omega_L = \omega_R = \omega$. The absorption rate is given by the difference of absorption and emission. These correspond to two different $i\epsilon$ prescriptions (Exercise: why?). So finally

$$\begin{aligned} \sigma_{abs} &\sim \Gamma_{abs} - \Gamma_{emit} && (13.33)\\ &\sim \int dtdxe^{-i\omega t}\left[G(t - i\epsilon, \phi) - G(t + i\epsilon, \phi)\right] && (13.34)\\ &\sim 2\frac{(2\pi T)^{2(h+\bar{h})-2}}{\Gamma(2h)\Gamma(2\bar{h})} \sinh\left(\frac{\omega}{2T}\right) \left|\Gamma(h + i\frac{\omega}{4\pi T})\Gamma(\bar{h} + i\frac{\omega}{4\pi T})\right|^2 && (13.35) \end{aligned}$$

This matches the gravity answer (13.25) if we set

$$h = \bar{h} = 1 \tag{13.36}$$

and use the identity $|\Gamma(1 + ix)|^2 = \pi x/\sinh(\pi x)$. Why should the weight be (13.36)? For now, we just pick them so the answer works out. In general the weights depend on the mass and spin of the bulk field, and (13.36) is the correct choice for a massless bulk field. We will treat this more systematically below.

13.4 Decoupling

The upshot of the last few sections is that

$$\begin{pmatrix} \text{Far-region gravity} \\ + \\ \text{gravity in } AdS_3 \times S^3 \end{pmatrix} \quad = \quad \begin{pmatrix} \text{Far-region gravity} \\ + \\ CFT_2 \text{ on } \ AdS_3 \text{ boundary} \end{pmatrix} .$$

In the gravity calculation, we assumed near-extremal but not exactly extremal. This retained some coupling between the near-horizon degrees of freedom, and the fields in the asymptotically flat far region. Similarly, in CFT, we assume a weak coupling between gravity fields and CFT fields.

If we take $T_H \to 0$, the far region and near regions decouple. This is Maldacena's decoupling limit. In this limit we can completely drop the asymptotically flat part of the calculation, and we are left with the (3d version of the) AdS/CFT correspondence:

$$\text{gravity in } AdS_3 \times S^3 \qquad = \qquad CFT_2 \text{ on } \ AdS_3 \text{ boundary} \ . \tag{13.37}$$

CFTs are UV-complete, so this duality defines not only low-energy effective gravity, but a UV-complete theory of gravity on $\mathrm{AdS}_3 \times S^3$.