12 Absorption Cross Sections of the D1-D5-P

We will now throw a scalar field at a near-extremal D1-D5-P black string. In the process we will rediscover AdS/CFT. The metric is (11.8), where we now assume

$$r_0 \ll r_1, r_5 \ . \tag{12.1}$$

This leads to a low Hawking temperature T_H. (We also assume $\cosh\sigma, r_1/r_5 \sim O(1)$.) Our goal is to calculate the absorption cross-section of a scalar field with low energy,

$$\omega r_5 \ll 1 \ . \tag{12.2}$$

We will assume the scalar χ has zero momentum around the ϕ direction and on the 3-sphere. It is convenient to define

$$T_L = \frac{1}{2\pi}\frac{r_0 e^{\sigma}}{r_1 r_5} \ , \quad T_R = \frac{1}{2\pi}\frac{r_0 e^{-\sigma}}{r_1 r_5} \ , \tag{12.3}$$

which will turn out to be left and right moving temperatures in the dual CFT. These are related to the Hawking temperature by

$$\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R} \ . \tag{12.4}$$

12.1 Gravity calculation

The wave equation $\Box\chi = 0$ for a scalar field of the form $\chi = e^{-i\omega t}R(r)$ in the metric (11.8) is

$$\left[\frac{h}{r^3}\frac{d}{dr}hr^3\frac{d}{dr} + \omega^2 f\right] R = 0 \tag{12.5}$$

where

$$f = \left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)\left(1 + \frac{r_0^2 \sinh^2\sigma}{r^2}\right) \ , \qquad h = 1 - \frac{r_0^2}{r^2} \ . \tag{12.6}$$

This is now basically a 1d quantum mechanics problem. To get some intuition for this scattering process, define

$$\chi(r) = \frac{1}{\sqrt{r(r^2 - r_0^2)}}\psi(r) \ . \tag{12.7}$$

In these variables, the wave equation is

$$\left[-\frac{d^2}{dr^2}+V(r)\right]\chi=0 \tag{12.8}$$

where $V(r)$ is easy to find and plot, but annoying to write down. It looks like a well near the horizon $r=r_0$, falls off at infinity, and has a lump somewhere in between. This looks just like an ordinary Schrodinger equation, so we are just scattering through a potential.

To compute the absorption cross-section, we need to solve the wave equation and compare the coefficients of the incoming, transmitted, and reflected waves. The strategy is to solve the equation approximately in the 'near' and 'far' regions, and match these solutions together somewhere in the middle. The near and far regions are defined by

$$\text{far:} \qquad r \gg r_0 \tag{12.9}$$

$$\text{near:} \qquad r \ll r_{1,5}, \qquad r \ll 1/\omega \tag{12.10}$$

These regions overlap in the 'matching region' $r_0 \ll r_m \ll r_{1,5}$.

The general solution of the wave equation in the far region is a linear combination of Bessel functions,

$$R_{far}=r^{-3/2}\sqrt{\frac{\pi\omega r}{2}}\left[AJ_1(\omega r)+BY_1(\omega r)\right] \tag{12.11}$$

The general solution in the near region is

$$R_{near}=\left[\tilde{A}h^{-i(a+b)/2}+\tilde{B}h^{+i(a+b)/2}\right]\,{}_2F_1(-ia,-ib,1-ia-ib,h) \tag{12.12}$$

with

$$a=\frac{\omega}{4\pi T_R}\ , \quad b=\frac{\omega}{4\pi T_L}\ . \tag{12.13}$$

The boundary condition is that the wave is purely ingoing at the horizon $r=r_0$. This sets $\tilde{B}=0$. Then we expand both R_{near} and R_{far} in the matching region:

$$R_{near} \approx \tilde{A}\frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\Gamma(1-ib)}+O(r_0^2/r^2) \tag{12.14}$$

$$R_{far} \approx A\frac{\sqrt{\pi}}{2\sqrt{2}}\omega^{3/2}+B\text{-terms} \tag{12.15}$$

We have not written the B terms because they are messy, but we will use conservation of flux to fix B later. Matching the terms in (12.14) gives

$$\sqrt{\frac{\pi\omega^3}{2}}\frac{A}{2} = \tilde{A}\frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\Gamma(1-ib)} \ . \tag{12.16}$$

The Wronskian of the 2nd order wave equation is interpreted as the conserved flux,

$$\mathcal{F} \equiv \frac{1}{2i}\left[hr^3 R^* \partial_r R - cc\right] , \qquad \frac{d\mathcal{F}}{dr} = 0 \ . \tag{12.17}$$

We would like to compare the incoming flux at infinity to the transmitted flux entering the horizon. The far solution, expanded near infinity, is

$$R_{far} \approx \frac{1}{2r^{3/2}}\left[e^{i\omega r}\left(Ae^{-3\pi i/4} - Be^{-i\pi/4}\right) + e^{-i\omega r}\left(Ae^{3i\pi/4} - Be^{i\pi/4}\right)\right] \ . \tag{12.18}$$

Thus the incoming flux is

$$\mathcal{F}_{in} = -\omega\left|\frac{A}{2}\right|^2 \ . \tag{12.19}$$

Using the same formula to calculate the flux through the horizon gives the absorbed flux

$$\mathcal{F}_{abs} = -r_0^2(a+b)|\tilde{A}|^2 \ . \tag{12.20}$$

The ratio of absorbed flux is (exercise!)

$$R_{abs} = \frac{\mathcal{F}_{ab}}{\mathcal{F}_{in}} = \frac{\omega^4\pi^2(r_1 r_5^2)}{4}\frac{e^{\omega/T_H}-1}{(e^{\omega/2T_L}-1)(e^{\omega/2T_R}-1)} \ . \tag{12.21}$$

Greybody factors

This is the greybody factor that appears in Hawking emission, up to a factor. The factor is required since the relation between spherical waves that we considered and plane waves is

$$e^{-i\omega z} = K\frac{e^{-i\omega r}}{r^{3/2}}Y_{000} + \cdots \tag{12.22}$$

where Y_{000} is the s-wave spherical harmonic on S^3. The constant is $K = \sqrt{4\pi/\omega^3}$. Therefore the absorption cross section for a plane wave is

$$\sigma_{abs} = |K|^2 R_{abs} \ . \tag{12.23}$$

This is the greybody factor.

Exercise: Far and near Hawking temperatures
Difficulty: straightforward
(a) Calculate the Hawking temperature of (11.8) (with $\sigma = 0$, *i.e.,* no rotation).

(b) Now calculate the Hawking temperature of the BTZ black hole that appears in the near horizon, (11.18) (again with $\sigma = 0$, so $w_- = 0$).

Note that when we took the near-horizon limit of the near-extremal string, we sent $r_0 \to 0$. So any finite temperature of the BTZ is actually *zero* temperature as viewed from asymptotically flat infinity. There is an infinite redshift between the near horizon region and infinity.

Exercise: Scattering of a massive scalar
Difficulty: difficult
The wave equation for a massive scalar is

$$\Box \chi = m^2 \chi \ . \tag{12.24}$$

In this problem we will derive the absorption cross section of a low-energy massive scalar on the near-extremal black string.

(a) Derive the full radial wave equation from (12.24), in the black string geometry (11.8) (but with $\sigma = 0$).

*(b)*Find the 'near-region' wave equation by starting with your answer to part (a) and assuming $r \ll r_{1,5}$ and $r\omega \ll 1$.

(c) Show that your near-region wave equation is identical to the massive wave equation on the BTZ $\times S^3$ geometry,

$$ds^2_{near} = \ell^2 \left[-(\tilde{r}^2 - r_0^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - r_0^2} + \tilde{r}^2 d\tilde{\phi}^2 + d\Omega_3^2 \right] \ . \tag{12.25}$$

(Where the tilded coordinates are proportional to the original coordinates.)

(c) Find the ingoing solution of the wave equation in the near region. Do not bother with the far-region solutions, since these are messy and nothing interesting happens in the far region.

Hint: Mathematica cannot solve this wave equation without some coaxing. To simplify it, first change variables so $h = 1 - r_0^2/r^2$ is your independent variable. Then define $R(h) = (1-h)^a h^b \psi(h)$, with $a = \frac{1}{2}(1 + \sqrt{1 + \ell^2 m^2})$ and $b = -i\omega\ell^2/2r_0$. Then Mathematica can solve it. This is also the best method to solve it by hand (*i.e.,* first strip out the singular points, then reduce the result to a standard hypergeometric equation).

(d) Show that in the matching region $r_0 \ll r \ll r_{1,5}$, the field behaves as

$$R_{near} \approx S r^{-d+\Delta} + F r^{-\Delta} \tag{12.26}$$

where $d = 2$,[58] S and F are numbers (possibly functions of ω), and

$$m^2 = \Delta(d - \Delta) \ , \qquad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2} \ . \tag{12.27}$$

(e) If we think of the S term as the source, or ingoing term, and F as the response, or outgoing term,[59] argue that the absorption cross section (of the near region) is proportional to the imaginary part of the ratio,

$$P_{abs} \equiv \mathrm{Im}\frac{F}{S} \tag{12.28}$$

and compute P_{abs}. (Including the far region too would just contribute some overall uninteresting factors.)

The correct answer looks like

$$P_{abs} = k \sinh(2\pi n) |\Gamma(\frac{\Delta}{2} + in)|^4 \tag{12.29}$$

[58] I've only written d so that the result is true in higher-dimensional $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$. In this problem always set $d = 2$.

[59] The words 'ingoing' and 'outgoing' are not quite accurate here since these are power-law solutions, not traveling waves, in the matching region. But they have a similar interpretation.

where $k = k(r_0, \Delta)$ is a simple constant you should find, and $n \equiv \ell^2\omega/(2r_0)$.

(f) Define the *retarded Green's function*

$$G_R = \frac{F}{S} \ . \tag{12.30}$$

This measures the response of the field to adding a source. (The relation (12.28) is a version of the optical theorem for this Green's function.)

Find G_R in the high-frequency limit $\omega/T_H \gg 1$. (This is the correlator in the extremal limit, where temperature goes to zero.) You should find a power law. This power-law behavior at short distances is the hallmark of a conformal field theory.[60]

(g) The zero-temperature 2pt function of a 2d CFT is

$$\langle O(x^+, x^-)O(0)\rangle = |x|^{-2\Delta} = (x^+x^-)^{-\Delta} \tag{12.31}$$

where Δ is the scaling dimension of the operator. Take the 2d Fourier transform,

$$G(\omega_L, \omega_R) \sim \int dx^+dx^- e^{i\omega_L x^+ + i\omega_R x^-}(x^+x^-)^{-\Delta} \ . \tag{12.32}$$

Don't worry about the coefficient; we only care about the power law, so you can do this Fourier transform by dimensional analysis. Check that for $\omega_L = \omega_R = \omega$, your answer agrees with part (f). Therefore, the quantity Δ that we introduced in the process of the solving the wave equation is equal to a CFT scaling dimension.

Exercise: Quasinormal modes

The scattering modes that we found above are modes that obey a single boundary condition: ingoing at the horizon. Such modes have a continuous spectrum. A *quasi-normal mode* is a mode that obeys two boundary conditions: ingoing at the horizon, and outgoing far away from the black hole. These have a discrete spectrum. They are *quasi*-normal instead of normal because they decay (as flux falls into the black hole) so the the discrete frequencies have imaginary parts. If you perturb a black hole from the vicinity of the horizon, the 'ringdown' is (roughly) described by quasinormal modes.

[60] Why did we have to take the high-energy limit to see this? The answer is that the temperature introduces a scale; correlators in a CFT are only scale-invariant in the vacuum.

The quasinormal modes of BTZ are modes which are ingoing at the horizon and have $S = 0$ in (12.26).

(a) Find the spectrum of of quasinormal modes ω_n for a massless scalar in BTZ.

(b) If you did the previous exercise, then use your solution of the near-region wave equation to find the spectrum of quasinormal modes ω_n for a massive scalar in BTZ.