11 AdS from Near Horizon Limits

Anti-de Sitter space appears in the near horizon region of extremal black holes. In this section we will describe how the near-horizon limit of extremal Reissner-Nordstrom is $\mathrm{AdS}_2 \times S^2$. Since the case $d = 1$ (*i.e.,* AdS_2) is a special case of AdS/CFT that we would like to mostly avoid, we then discuss the 6d black string. This solution has a near-horizon AdS_3 which will serve as our main example for AdS/CFT.

11.1 Near horizon limit of Reissner-Nordstrom

Here is the 4d Reissner-Nordstrom black hole again, from (2.10):

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \ , \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \ . \tag{11.1}$$

Recall that this solution is restricted to $M > Q$ (we assume $Q > 0$) by cosmic censorship. In general, the near horizon geometry is approximately Rindler $\times S^2$, as discussed above. But something special happens in the *extremal limit*,

$$M = Q \ . \tag{11.2}$$

In this limit the horizon is a double zero, $f(r) = (1 - Q/r)^2$. So the inner and outer horizons coincide, $r_+ = r_- = Q$, and the Hawking temperature (2.19) is zero.

To take the near horizon limit of the extremal black hole, we define

$$r = Q(1 + \lambda/z), \quad t = QT/\lambda \tag{11.3}$$

where λ is an arbitrary parameter. Plugging this into the extremal metric and taking the limit $\lambda \to 0$ with z, T, θ, ϕ held fixed gives the spacetime

$$ds^2 = \frac{Q^2}{z^2}\left(-dT^2 + dz^2\right) + Q^2 d\Omega_2^2 \ . \tag{11.4}$$

The metric (11.4) is $AdS_2 \times S^2$. This is the near horizon region of the original black hole, since if $\lambda \to 0$ with z held fixed, $r \to Q$. Recall that this spacetime is supported by some nontrivial electric field. Applying the same procedure to the field strength gives a constant electric field in the near horizon region.[56]

λ has disappeared entirely from the solution. This means that (11.4) (together with the constant electric field) is actually by itself a solution to the Einstein-Maxwell equations. This did not happen for the non-extremal case: there, the spacetime was only approximately Rindler near the horizon, and Rindler $\times S^2$ does not solve the Einstein-Maxwell equations.

Geometry of the Near Horizon Region

The key difference between the extremal and non-extremal near horizon limits is that *the near horizon region of an extremal black hole is infinitely long.* To see this let us calculate the distance to the horizon along a fixed-t slice in the general metric (11.1), from some arbitrary point $r_0 > r_+$:

$$D = \int_{r_+}^{r_0} dr \sqrt{f(r)} \sim -M \log(r_+ - r_-) \sim M \log \frac{1}{QT_H} \tag{11.5}$$

where T_H is the Hawking temperature (2.19). This diverges as $T_H \to 0$. (The $\sim$ means we are dropping constants and the contribution of the r_0 limit to the integral, which doesn't matter.) The long region for small T_H

Global coordinates

The near-horizon metric we found in (11.4) is AdS_2 in Poincaré coordinates. This covers only a patch of the full AdS spacetime. Similarly, we considered only one patch of the Reissner-Nordstrom spacetime. The full global AdS_2 includes many Poincaré patches, and each patch gives the near-horizon region of a different patch of the global Reissner-Nordstrom. This is illustrated in the Penrose diagrams in figure 4.

What about the sphere?

We are only drawing the Penrose diagrams for AdS_2, but the geometry is in fact $\mathrm{AdS}_2 \times S^2$. Actually the sphere does not affect the conformal boundary. Since the

[56]In the original coordinates, $F_{rt} = Q/r^2$. In the new coordinates, after scaling $\lambda \to 0$, $F_{zT} = -Q/z^2$. This does not look constant in these coordinates, but if we define $\sigma = 1/z$ then $F_{\sigma T} = Q$ is a constant electric field.

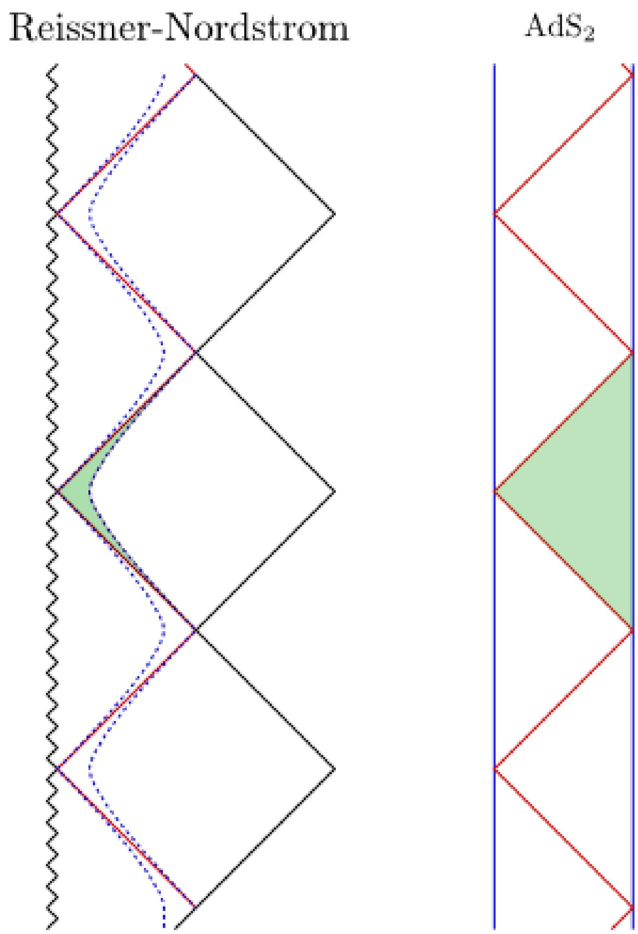

Figure 4: Penrose diagrams for extremal Reissner-Nordstrom and AdS_2. The AdS_2 Penrose diagram is a 'zoomed in' version of the RN diagram which includes only the near horizon region. Unlike higher-dimensional AdS, AdS_2 has two conformal boundaries, which are the blue lines on the left and right. In the RN diagram, the coordinates are degenerate near the horizon so these boundaries are drawn as dashed blue lines slightly away from $r = r_+$. The black hole horizon in RN (red) is the same as the Poincaré horizon in AdS_2. The Poincaré patch of AdS_2 is shaded green in both diagrams.

prefactor in front of AdS_2 blow up near the boundary, after a conformal rescaling the sphere just drops out. So points on the conformal boundary are labeled only by T, not by T, θ, ϕ.

Near horizon as a low-energy limit

A particle on a geodesic has a conserved energy-at-infinity

$$E \equiv -p_t = f(r) \frac{dt}{d\tau} \ . \tag{11.6}$$

For a particle in the near-horizon region, this is strictly zero as $\lambda \to 0$. So from the point of view of an observer at infinity, everything in the near-horizon region is infinitely redshifted. Similarly, a wave in the near-horizon region $\propto e^{-i\omega_{near} T}$ has zero frequency as measured from infinity, since

$$\omega_{near} T \sim \lambda \omega t \sim 0 \cdot t \ . \tag{11.7}$$

11.2 6d black string

Unfortunately AdS_2 is the runt of the AdS/CFT litter. It is very interesting in its own right but quite different from other dimensions (since CFT_1 does not really make sense) so not suitable for our purposes. We will focus on AdS_3 instead, which appears in the near horizon limit of a 6d black string (among other things). In the exercises you will treat the other most popular example of AdS/CFT which involves AdS_5.

The 6d black string is similar to a black hole, but with horizon topology $S^3 \times S^1$. The 6d metric is[57]

$$\begin{aligned} ds^2 = (f_1 f_5)^{-1/2} &\left(-dt^2 + d\phi^2 + \frac{r_0^2}{r^2} (\cosh \sigma dt + \sinh \sigma d\phi)^2 \right) \\ &+ (f_1 f_5)^{1/2} \left(\frac{dr^2}{1 - r_0^2/r^2} + r^2 d\Omega_3^2 \right) \ . \end{aligned} \tag{11.8}$$

[57]My favorite references on this solution and the discussion to follow are Kiritsis's textbook, section 12.7, and MAGOO hep-th/9905111. Note that I will not distinguish between the Einstein-frame metric and string-frame metric in this discussion; they differ by just a constant in the near-horizon region which can be absorbed into the definition of ℓ.

where

$$f_1 = 1 + \frac{r_1^2}{r^2}, \qquad f_5 = 1 + \frac{r_5^2}{r^2} \ , \tag{11.9}$$

and $\phi \sim \phi + 2\pi R$ is compact. I will not write the other fields but there are some nontrivial scalars and gauge fields that can be found in the references. This solution carries two charges, called Q_1 and Q_5, related to r_1 and r_5. (In string theory these count the number of D1 branes and D5 branes.) It also carries a momentum proportional to $r_0 \sinh\sigma$ along the ϕ direction, as you can guess from the boost term $\cosh\sigma dt + \sinh\sigma d\phi$. Finally r_0 is the position of the horizon and measures the deviation from extremality. To see this, note the surface gravity is proportional to $\partial_r g_{rr}^{-1}|_{r=r_0}$.

This is called the D1-D5-P black string. (Often in the literature you will find that it is dimensionally reduced to 5d along the ϕ direction, so it becomes a 3-charge black hole.)

AdS$_3$ in the near horizon of the extremal black string

Far away, *i.e.,* $r \gg 0$, this geometry is just $R^4 \times S^1$. Now we will look at the decoupling near horizon limit. This the same type of near horizon limit that we did for extremal RN, so it produces an exact, infinite-volume solution of the equations of motion.

First we consider the case with zero momentum. The extremal D1-D5 with $P = 0$ is obtained by setting $r_0 = 0$,

$$ds^2 = (f_1 f_5)^{-1/2}(-dt^2 + d\phi^2) + (f_1 f_5)^{1/2}(dr^2 + r^2 d\Omega_3^2) \ . \tag{11.10}$$

The horizon is at $r = 0$. To take the near-horizon limit, we define

$$\ell^2 = r_1 r_5 \tag{11.11}$$

scale

$$r \to \lambda \ell r \ , \quad t \to t\ell/\lambda \ , \quad \phi \to \phi \ell/\lambda \ , \tag{11.12}$$

and send $\lambda \to 0$. This has the effect of just dropping the 1 in $f_i = 1 + r_i^2/r^2$, so

$$ds^2_{near} = \ell^2 \left(\frac{dr^2}{r^2} + r^2(-dt^2 + d\phi^2) \right) + \ell^2 d\Omega_3^2 \ . \tag{11.13}$$

This is the geometry $\mathrm{AdS}_3 \times S^3$, where the curvature radii of AdS and of the sphere are equal.

Near-extremal D1-D5-P

Now let us take a different near-horizon limit of (11.8) where we simultaneously scale $r \to 0$ as we scale the black string towards extremality, $r_0 \to 0$. In this limit, $r_0 \cosh\sigma$ stays finite, so this is an extremal limit with finite momentum.

Starting from (11.8) we scale

$$r \to \lambda\ell r \ , t \to t\ell/\lambda \ , \phi \to \phi\ell/\lambda \ , r_0 \to \lambda\ell r_0 \ . \tag{11.14}$$

The resulting metric is

$$ds^2_{near} = \ell^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2 - r_0^2} + r^2 d\phi^2 + r_0^2 \left(\cosh\sigma dt + \sinh\sigma d\phi\right)^2 \right] + \ell^2 d\Omega_3^2 \ . \tag{11.15}$$

The term in brackets is in fact a 3d black hole, called the BTZ black hole. To see this in more standard BTZ coordinates, define the parameters

$$w_+ = r_0 \cosh\sigma, \qquad w_- = r_0 \sinh\sigma \tag{11.16}$$

and do the coordinate change

$$r^2 = w^2 - w_-^2 \ . \tag{11.17}$$

The resulting metric is

$$\frac{ds^2_{near}}{\ell^2} = -h(w)dt^2 + \frac{dw^2}{h(w)} + w^2 \left(d\phi + \frac{w_+ w_-}{w^2} dt\right)^2 + d\Omega_3^2 \ , \tag{11.18}$$

where

$$h(w) = \frac{(w^2 - w_+^2)(w^2 - w_-^2)}{w^2} \ . \tag{11.19}$$

The w, t, ϕ part of this metric is a 3d black hole carrying mass and angular momentum, with horizons at $w_\pm$. Setting $w_- = 0$ and $w_+ = 8M$ gives the $J = 0$ BTZ that was used in some examples earlier in the course.

Exercise: AdS_5 as near horizon limit

Difficulty: easy

Consider the 10D metric

$$ds^2 = f^{-1/2}(-dt^2 + d\vec{x}^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \ . \tag{11.20}$$

where

$$f = 1 + \frac{r_3^4}{r^4} \ . \tag{11.21}$$

r_3 a constant parameter and $\vec{x}$ a coordinate on 4d space R^4. This metric is an extremal black brane. (A *black brane* is like a black hole, but the horizon is a plane instead of a sphere. In string theory, this solution is the geometry corresponding to a stack of Q_3 D3 branes, where Q_3 is a conserved charge of this solution, related to r_3.)

Show that the near-horizon geometry is $\mathrm{AdS}_5 \times S^5$.

Exercise: Near horizon Kerr

Difficulty: A little messier, use Mathematica!

The metric of the 4D Kerr black hole is

$$ds^2 = -\frac{\Delta(r)}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta(r)}dr^2 + \rho^2 d\theta^2 + \frac{1}{\rho^2}\sin^2\theta(adt - (r^2+a^2)d\phi)^2 \ , \tag{11.22}$$

where

$$\Delta(r) = r^2 + a^2 - 2Mr \ , \qquad \rho^2 = r^2 + a^2\cos^2\theta \ , \tag{11.23}$$

and $-M < a < M$. This describes a rotating black hole with mass M and angular momentum

$$J = aM \ . \tag{11.24}$$

Find the value of M (as a function of J) where this black hole is extremal. Then find the near horizon geometry of the extremal Kerr.

Hint: To find a regular near-horizon metric, you must go to a rotating coordinate system that corotates with the black hole horizon, $\psi = \phi - \Omega t$. After going to this rotating coordinate system the calculation is similar to what we did for RN.

Reference: Bardeen and Horowitz, [hep-th/9905099]. (Also [0809.4266].)