10 Interlude: Preview of the AdS/CFT correspondence

The rest of this course is, roughly speaking, on the AdS/CFT correspondence, also known as 'holography' or 'gauge/gravity duality' or various permutations of these words. AdS/CFT was conjectured by Maldacena in a famous paper in 1997. A full understanding of Maldacena's motivations and results, and the huge body of work to follow, requires some string theory, but AdS/CFT itself is independent of string theory and we will not follow this route. Instead we will 'discover' AdS/CFT by throwing stuff at black holes. In fact, this parallels the historical discovery of AdS/CFT in 1996-1997, though we will obviously take a shorter path. Our starting point will be a black-hole-like solution in 6 dimensions, which might seem umotivated, so the purpose of this interlude is to describe where we are headed, so you know we are doing this for a good reason.

10.1 AdS geometry

Anti-de Sitter space is the maximally symmetric solution of the Einstein equations with negative cosmological constant. We worked out the metric of AdS_3 in global and Poincaré coordinates in the previous section. For general dimension AdS_{d+1}, the metric in global coordinates is

$$ds^2 = \ell^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2 \right) \ . \tag{10.1}$$

To find the Penrose diagram, we can extract a factor of $\cosh^2 \rho$ and then define a new coordinate by

$$d\sigma = \frac{d\rho}{\cosh \rho} \Rightarrow \sigma = 2 \tan^{-1} \tanh(\rho/2) \ . \tag{10.2}$$

As ρ runs from 0 to ∞, ρ runs from 0 to $\pi/2$. Each value of t, σ is a sphere S^{d-1}. Therefore the Penrose diagram looks like a solid cylinder, where ρ is the radial coordinate of the cylinder, and t, Ω are the coordinates on the surface of the cylinder.

Unlike flat space, the conformal boundary (usually just called 'the boundary') of AdS is *timelike*. From the Penrose diagram, we can see that massless particles reach the boundary in finite time t. (Massive particles cannot reach the boundary; they feel an $e^{2\rho}$ potential if they try to head to large ρ.)

The metric of the Poincaré patch is

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right) \ , \tag{10.3}$$

where $\vec{x} = (x^1, \ldots, x^{d-1})$. In these coordinates, the boundary is at $z = 0$. These coordinates cover a wedge of the global cylinder. You can check this for AdS_3 using the coordinate transformations derived in the previous section.

10.2 Conformal field theory

A conformal field theory (CFT) is a QFT with a particular spacetime symmetry, *conformal invariance*. Conformal invariance is a symmetry under local scale transformations. We will discuss this in detail later. For now I will just mention that one consequence of conformal symmetry is that correlation functions behave nicely under coordinate rescalings $x \to \lambda x$. Correlation functions of primary operators (which are lowest weight states of a conformal representation) obey

$$\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n)\rangle = \lambda^{\Delta_1+\Delta_2+\cdots+\Delta_n}\langle O_1(\lambda x_1)\cdots O_n(\lambda x_n)\rangle \tag{10.4}$$

where Δ_i is called the *scaling dimension* of the operator O_i. This (together with rotation and translation invariance) implies for 2pt functions

$$\langle O(x) O(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}} \ . \tag{10.5}$$

The simplest example of a CFT is a free massless scalar field, where for example in 4d $\langle\phi(x)\phi(y)\rangle = (x-y)^{-2}$. A massive free field is not conformal, since m shows up in correlation functions and spoils the simple power behavior. This is generally true – CFTs do not have any dimensionful parameters, so there can be no mass terms in the Lagrangian. However the converse is not true, since there are theories with no mass terms in the classical theory are not necessarily conformal. For example in massless QCD, scale symmetry is broken in the the quantum theory so the theory acquires a dimensionful parameter via dimensional transmutation.

There are also very nontrivial interacting conformal field theories. We will discuss a couple of examples later.

10.3 Statement of the AdS/CFT correspondence

The AdS/CFT correspondence is the an exact relationship between any[54] theory of quantum gravity in asymptotically AdS_{d+1} spacetime and an ordinary CFT_d, without gravity. This relationship is called a *duality*. It is *holographic* since the gravitational theory lives in (at least) one extra dimension. The theories are believed to be entirely equivalent: any physical (gauge-invariant) quantity that can be computed in one theory can also be computed in the dual. However, the mapping between the two theories can be highly nontrivial. For example, easy calculations on one side often map to strongly coupled, incalculable quantities on the other side.

It is often useful to think of the CFT as 'living at the conformal boundary' of AdS. Indeed, the CFT lives in a spacetime parameterized by $x = (t, \vec{x})$, whereas gravity fields are functions of x and the radial coordinate ρ. And when we discuss correlation functions of local operators we will see that a CFT point x corresponds to a point on the conformal boundary of AdS. But it is not quite accurate to say that the CFT lives on the boundary, for two reasons. First, we should not think about having both theories at once; we either do CFT or we have an AdS spacetime, never both at the same time. Second, the CFT is dual to the entire gravity theory, so in a sense it lives everywhere.

The two theories are commonly referred to as 'the bulk' (*i.e.,* the gravity theory) and 'the boundary' (ie the CFT).

In this course we will mostly restrict our attention to two types of observables in AdS/CFT: thermodynamic quantities and correlation functions.

[54] Some people might obect to the word 'any' here. To be safe, we could say 'any theory that we know how to define in the UV and acts like ordinary gravity+QFT in the IR.'

Thermodynamics

The mapping between thermodynamic quantities on the two sides of the duality is simply that they should be equal, for example the thermal partition functions obey

$$Z_{cft}(\beta) = Z_{gravity}(\beta) \ . \tag{10.6}$$

Here $Z_{cft} = \text{ Tr } e^{-\beta H_{cft}}$ is the ordinary thermodynamic partition function of a QFT. $Z_{gravity}$ is the quantity whose semiclassical limit we discussed above, related to the on-shell action of a black hole,

$$Z_{gravity}(\beta) = e^{-S_E[g]} + \cdots \ . \tag{10.7}$$

For the exact relation (10.6) we must in principle include all the quantum corrections to this semiclassical formula.

Correlation functions

The goal of the next couple lectures is to derive the *dictionary* that relates CFT correlators to a gravity calculation. We will give the exact prescription later, but here is the general idea. Each field $\phi_i(\rho, x)$ in the gravitational theory there is a corresponding operator $O_i(x)$ in the CFT.[55] The mass of ϕ determines the dimension of O. CFT correlation functions can be computed on the gravity side by computing a gravity correlator of ϕ, with the points inserted at the boundary:

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle_{cft} \leftrightarrow \text{``}\lim_{\rho\to\infty}\text{''}\langle \phi_1(\rho, x_1)\cdots\phi_n(\rho, x_n)\rangle_{gravity} \tag{10.8}$$

The limit is in quotes because actually we need to rescale by some divergent factors that we'll come to later.

Top down, bottom up, and somewhere in between

AdS/CFT is general, we do not need to refer to a particular theory of a gravity or a particular CFT. However it is often useful to have specific theories in mind, with

[55]Here $x = (t, \vec{x})$ denotes all d dimensions of the CFT and ρ is the radial coordinate.

detailed microscopic definitions. For example: *Type IIB string theory on* $AdS_5 \times S^5$ *is dual to* $\mathcal{N} = 4$ *supersymmetric Yang-Mills in 4d.*

Super-Yang-Mills is a particular CFT with a known Lagrangian. Although IIB string theory is not defined non-perturbatively (except via this duality), it has many known microscopic ingredients. Calculations in these two specific theories can be compared in great detail.

There are other microscopic examples — deformations of this one, and different versions in different dimensions, with different types of dual CFTs. All of them (as far as I know) come from brane constructions in string theory. This is often called the 'top down' approach to AdS/CFT.

Another approach is to simply assume that we have a CFT with some low-dimension primaries with a particular pattern, and perhaps with some assumptions about the symmetries and conserved charges of the theory. This is more in the spirit of effective field theory and is often called 'bottom up.' In many cases we can also include information about the UV completion of the CFT (*i.e.,* very high dimension operators) in this approach so it actually goes beyond effective field theory, but without every specifying the actual Lagrangian of the CFT.

Both approaches are important. Often calculations that can be done in one approach are impossible in the other, or calculations first done microscopically turn out to have more general and possibly more intuitive explanations via effective field theory.